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ON TRIPLETS OF PLANE CURVILINEAR ELEMENTS WITH A COMMON SINGULAR POINT

By C. C. HSIUNG (*Chekiang*)

[Received 21 April 1946]

1. Introduction

It is known that, if two plane curves have a point O for their common singular point of the same kind with the same tangent, so that the power-series expansions for non-homogeneous projective coordinates representing these two curves in the neighbourhood of O are

$$y = ax^k + \dots \quad \text{and} \quad y = bx^k + \dots \quad (a \neq b),$$

then the ratio a/b is a projective invariant of the two curves. C. Segre* gave a simple geometrical characterization of this invariant for any value of k. About four decades later E. Bompiani† interpreted it again, by using certain algebraic curves, for the case in which k is a fraction. Since no corresponding projective invariant can be determined for two curves having a common singular point of different kinds, it is desirable to undertake the investigation for three curves. This is the object of the present paper.

2. Derivation of an invariant

Suppose that C_1, C_2, C_3 are three plane curves having O as a common singular point of different kinds but with the same tangent, so that the behaviours of the common tangent t at the point O with them are distinct. Let the homogeneous projective coordinates of a point in the plane be denoted by (x, y, z). If we choose the point O to be the vertex $(0, 0, 1)$ and the common tangent t to be the side $y = 0$ of the triangle of reference, then the power-series expansions of the curves in the neighbourhood of O can be written in the forms

$$C_1\colon\quad \frac{y}{z} = a\left(\frac{x}{z}\right)^{\lambda} + \dots, \tag{1}$$

$$C_2\colon\quad \frac{y}{z} = b\left(\frac{x}{z}\right)^{\mu} + \dots, \tag{2}$$

$$C_3\colon\quad \frac{y}{z} = c\left(\frac{x}{z}\right)^{\nu} + \dots. \tag{3}$$

* C. Segre, 'Su alcuni punti singolari delle curve algebriche, e sulla linea parabolica di una superficie', *Rendi. dei Lincei* (5), 6 (1897), 168–75.

† E. Bompiani, 'Alcuni risultati di geometria proiettivo-differenziale', *Rendi. del Semin. Mat. e Fis. di Milano*, 10 (1936), 1–28.

Without loss of generality we may assume the exponents λ, μ, ν in (1), (2), (3) to be distinct integers or fractions greater than 1.

In order to find a projective invariant associated with the singular point O of the curves C_1, C_2, C_3, I consider the most general projective transformation of coordinates that leaves invariant the point O and the common tangent t:

$$x = a_{11}x'+a_{12}y', \qquad y = a_{22}y', \qquad z = a_{31}x'+a_{32}y'+a_{33}z'. \tag{4}$$

The effect of this transformation on equations (1), (2), (3) is to produce three other equations of the same form whose coefficients, indicated by accents, are given by the formulae

$$a_{22}a_{33}^{\lambda-1}a' = a_{11}^{\lambda}a, \qquad a_{22}a_{33}^{\mu-1}b' = a_{11}^{\mu}b, \qquad a_{22}a_{33}^{\nu-1}c' = a_{11}^{\nu}c. \tag{5}$$

Further, elimination of a_{11}, a_{22}, a_{33} from equations (5) shows immediately that *the expression*

$$I = a^{\mu-\nu}b^{\nu-\lambda}c^{\lambda-\mu} \tag{6}$$

is a projective invariant associated with the singular point O of the curves C_1, C_2, C_3.

3. Geometrical characterizations of the invariant I.

In this section I shall characterize geometrically the invariant I obtained in the previous section. It is convenient to distinguish four cases, according as λ, μ, ν are integers or fractions.

Case I. All three exponents λ, μ, ν integers. Let the vertices (1, 0, 0), (0, 1, 0) of the triangle of reference be denoted by O_1, O_2 respectively. It is easily seen that, associated with the singular point O of the curve C_1, we can uniquely determine an algebraic curve of order λ having a $(\lambda-1)$-ple point at O_2 with all tangents coinciding in the line O_1O_2, and having a contact of order λ with the curve C_1 at O. From (1), the equation of this curve is readily found to be

$$yz^{\lambda-1}-ax^{\lambda} = 0. \tag{7}$$

From (2), (3), we can likewise obtain the equations of two analogous algebraic curves of orders μ, ν associated with the singular point O of the curves C_2, C_3,

i.e.

$$yz^{\mu-1}-bx^{\mu} = 0, \tag{8}$$

$$yz^{\nu-1}-cx^{\nu} = 0. \tag{9}$$

Elimination of z from (7), (8) yields the equation of the line joining the point O to any intersection, other than O, of the curves (7), (8):

$$x-\left(\frac{a^{\mu-1}}{b^{\lambda-1}}\right)^{1/(\lambda-\mu)}y = 0. \tag{10}$$

Likewise, the line joining the point O to any intersection, other than O, of the curves (7), (9) is given by the equation

$$x-\left(\frac{a^{\nu-1}}{c^{\lambda-1}}\right)^{1/(\lambda-\nu)}y = 0. \tag{11}$$

Thus we obtain the following geometrical characterization:

The cross-ratio of the line OO_2, the common tangent t, and the two lines (10), (11) *is equal to*

$$\left(0, \infty, -\left(\frac{a^{\mu-1}}{b^{\lambda-1}}\right)^{1/(\lambda-\mu)}, -\left(\frac{a^{\nu-1}}{c^{\lambda-1}}\right)^{1/(\lambda-\nu)}\right) = I^{(\lambda-1)/(\lambda-\mu)(\lambda-\nu)}. \tag{12}$$

Case II. All three exponents λ, μ, ν fractions. For this case we may put

$$\lambda = \frac{\lambda_1}{\lambda_2}, \qquad \mu = \frac{\mu_1}{\mu_2}, \qquad \nu = \frac{\nu_1}{\nu_2}, \tag{13}$$

where $\lambda_1 > \lambda_2$, $\mu_1 > \mu_2$, $\nu_1 > \nu_2$, and (λ_1, λ_2), (μ_1, μ_2), (ν_1, ν_2) are three pairs of relatively prime integers. First of all we observe that, associated with the singular point O of the curve C_1, we can uniquely determine an algebraic curve of order λ_1 satisfying the following conditions:

(i) its polar curves of O_2 of orders

$$1,..., \lambda_1-\lambda_2-1, \lambda_1-\lambda_2+1,..., \lambda_1-1$$

are indeterminate, while that of order $\lambda_1-\lambda_2$ degenerates into the line $O_1 O_2$ counted $\lambda_1-\lambda_2$ times;

(ii) O is a singular point at which the curve has contact of order λ_1-1 with t;

(iii) the curve has contact of order λ_1 with C_1 at O.

From $\lambda = \lambda_1/\lambda_2$ and (1), it is easy to obtain the equation of this algebraic curve, namely

$$y^{\lambda_2} z^{\lambda_1-\lambda_2} - a^{\lambda_2} x^{\lambda_1} = 0. \tag{14}$$

If the curves C_2, C_3 are used instead of C_1, two similar algebraic curves of orders μ_1, ν_1 can be determined whose respective equations are

$$y^{\mu_2} z^{\mu_1-\mu_2} - b^{\mu_2} x^{\mu_1} = 0, \tag{15}$$

$$y^{\nu_2} z^{\nu_1-\nu_2} - c^{\nu_2} x^{\nu_1} = 0. \tag{16}$$

Elimination of z from (14), (15) shows immediately that the equation of the line l joining O to any intersection, other than O, of the curves

(14), (15) can be written as (10). Similarly, the line l' joining O to any intersection, other than O, of the curves (14), (16) is given by (11). Thus we arrive at the following geometrical characterization of the invariant I for this case:

The cross-ratio of the line OO_2, the common tangent t, and the two lines l, l' above is equal to $I^{(\lambda-1)(\lambda-\mu)(\lambda-\nu)}$.

Case III. Two only of the exponents λ, μ, ν are integers. Let λ, μ be integers and ν a fraction denoted by ν_1/ν_2 as in the previous case. Now the three curves (7), (8), (16) can be used for our purpose. If p be the line joining the point O to any intersection, other than O, of the curves (7), (16), then *the cross-ratio of the line OO_2, the common tangent t, and the two lines* (10), *p is equal to* $I^{(\lambda-1)(\lambda-\mu)(\lambda-\nu)}$.

Case IV. Only one of the exponents λ, μ, ν is an integer. Let λ be an integer and μ, ν fractions denoted by μ_1/μ_2, ν_1/ν_2 as in Case I. If q, q' be the lines joining O to two intersections, other than O, of the curve (7) with the curve (15) and with the curve (16) respectively, then *the cross-ratio of the line OO_2, the common tangent t, and the two lines q, q' is equal to* $I^{(\lambda-1)(\lambda-\mu)(\lambda-\nu)}$.

ON SYSTEMS OF ALGEBRAIC EQUATIONS AND CERTAIN MULTIPLE EXPONENTIAL SUMS

By S. H. MIN (*Peiping*)

[Received 10 July 1946]

1. THROUGHOUT the paper we use p to denote a prime. Let

$$f(x) = a_k x^k + ... + a_1 x \quad (k \geqslant 2),$$

be a polynomial with integer coefficients. The exponential sum

$$S\{f(x)\} = \sum_{x=1}^{p} e^{2\pi i f(x)/p},$$

is called a *generalized Gaussian sum*. Mordell proved that, if a_k is not divisible by p, then, as p tends to ∞,

$$S\{f(x)\} = O(p^{1-1/k}),$$

where the constant implied by O depends only on k.

The n-dimensional generalization of $S\{f(x)\}$ is the n-ple exponential sum

$$S\{f(x_1,..., x_n)\} = \sum_{x_1=1}^{p} ... \sum_{x_n=1}^{p} \exp\left\{\frac{2\pi i}{p} f(x_1,..., x_n)\right\},$$

where $f(x_1,..., x_n)$ is a polynomial of degree k with integer coefficients. More generally we consider a field K containing p^m elements and suppose that $f(x_1,..., x_n)$ is a polynomial of degree k in the field K. Construct the sum

$$S\{f(x_1,..., x_n), K\} = \sum_{x_1} ... \sum_{x_n} \exp\left\{\frac{2\pi i}{p} \mathfrak{S}[f(x_1,..., x_n)]\right\},$$

where $\mathfrak{S}[a]$ is the trace of a, and $x_1,..., x_n$ run respectively over all elements in K.

Now let

$$f(\alpha_1 t+\beta_1,..., \alpha_n t+\beta_n) = \sum_{r=0}^{k} F_r(\alpha_1,..., \alpha_n; \beta_1,..., \beta_n) t^r$$

and

$$\mathfrak{M} = \begin{bmatrix} \frac{\partial F_1}{\partial \alpha_1} & . & . & . & \frac{\partial F_1}{\partial \alpha_n} & \frac{\partial F_1}{\partial \beta_1} & . & . & . & \frac{\partial F_1}{\partial \beta_n} \\ . & . & . & . & . & . & . & . & . & . \\ \frac{\partial F_k}{\partial \alpha_1} & . & . & . & \frac{\partial F_k}{\partial \alpha_n} & \frac{\partial F_k}{\partial \beta_1} & . & . & . & \frac{\partial F_k}{\partial \beta_n} \end{bmatrix}.$$

I shall prove that, if $k \geqslant 2n$ and $\mathfrak{M}$ is of rank $2n$, i.e. if a certain $2n \times 2n$ minor is not identically zero, then, as p tends to infinity,

$$S\{f(x_1,..., x_n), K\} = O\{p^{mn(1-1/k)}\},$$

where the constant implied by O depends only on k and n.

When $m = n = 1$, $\mathfrak{M}$ is of rank 2 unless $f(x_1)$ reduces to a linear function of x_1, so that the theorem includes Mordell's theorem as a special case. In fact, when $n = 1$, the minor determinant formed by the first two rows is

$$\tfrac{1}{2}\alpha_1^2[f'(\beta_1)f'''(\beta_1)-2\{f''(\beta_1)\}^2].$$

This is identically zero only when $f(x_1)$ is a linear polynomial.

When $n = 2$, the theorem was established by Hua and me. We also proved by an elaborate method* that, if $\mathfrak{M}$ is of rank $< 2n = 4$, $f(x_1, x_2)$ can be transformed by a linear non-singular transformation in K to $g(x, y)$ which is either of degree $\leqslant \frac{1}{2}k$ in y or is a polynomial in $x^2+\tau y^2$ $(\tau \in K)$. From this we deduced that the theorem holds unless† $f(x_1, x_2)$ is a polynomial in a single variable which is itself a linear function of x and y. It is very likely that a similar result exists in the general case, but this seems to be very difficult to establish. In the two-dimensional case I have obtained a still better result, which, roughly speaking, may be regarded as a two-dimensional analogue of a result due to Davenport.‡ I hope to publish it elsewhere.

2. In the theory of a system of linear equations the most important role is played by the matrix of the system. A natural generalization of this is the 'Jacobian matrix' of a system of algebraic equations.

Let $$f_i(x_1,..., x_n) = 0 \quad (i = 1,..., m) \tag{1}$$

* 'On a double exponential sum', *Science Report of Tsing-Hua Univ.* (in the press). A short sketch of the proof has been published in the *Science Record of Academia Sinica*, vol. i, Nos. 1–2. It is impossible, however, to give a sufficiently complete short sketch.

† The method of procedure where $f(x_1, x_2)$ is a polynomial in $x^2+\tau y^2$ may be outlined as follows (taking $m = 1$ for simplicity). The sum is the same as the corresponding sum with $x^2+\tau y^2$ replaced by $x+\tau y$, provided that each term is multiplied by

$$\{1+(x/p)\}\{1+(y/p)\}.$$

This gives four sums; only the one containing (xy/p) presents difficulty. This is dealt with by putting $y = xz$, and applying Mordell's theorem to the resulting sum over x.

‡ 'On certain exponential sums', *J. für die reine und angew. Math.*, 169, Heft 3 (1933), 158–76.

be a system of algebraic equations in a field F. We write the Jacobian matrix as

$$\mathfrak{J}(f_1,\dots,f_m) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & . & . & . & \dfrac{\partial f_1}{\partial x_n} \\ . & . & . & . & . \\ \dfrac{\partial f_m}{\partial x_1} & . & . & . & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{2}$$

and, when $m = n$, denote its determinant by $J(f_1,\dots,f_m)$.

DEFINITION. *A solution of* (1) *for which the matrix* $\mathfrak{J}(f_i,\dots,f_m)$ *is of rank* r *is called a solution of rank* r. *If* $r = n$, *the solution is said to be non-singular; otherwise it is singular.*

The theorems to be proved hold both for fields of characteristic zero and those of characteristic p. We are interested in large primes p and polynomials of fixed degree.

Corresponding to the fact that a system of linear equations has only one solution if its determinant is different from zero, is the following

THEOREM 1. *The number of non-singular solutions of* (1) *is* $O(1)$, *where the constant implied by* O *depends only on* n, m *and the degrees of* f_i ($i = 1,\dots,m$).

Proof. Without loss of generality we may assume that $m = n$. The theorem is evident for $n = 1$. Suppose that the theorem is true for $n-1$ variables and $n-1$ equations. Let us prove the theorem for n variables and n equations.

If $f_1,\dots,f_n$ do not contain* x_n, the theorem is true, for in this case $J(f_1,\dots,f_n) = 0$ identically and (1) has no non-singular solutions. If only one of them contains x_n the theorem is also true; for supposing that f_n contains x_n and that $(x_1^{(0)},\dots,x_{n-1}^{(0)},x_n^{(0)})$ is a non-singular solution of (1), then we have

$$J(f_1,\dots,f_n) = \frac{\partial f_n}{\partial x_n} J(f_1,\dots,f_{n-1}) \neq 0 \quad \text{at } x = x_\nu^{(0)},$$

which shows that $(x_1^{(0)},\dots,x_{n-1}^{(0)})$ is a non-singular solution of $f_1 = 0,\dots, f_{n-1} = 0$ and that $f_n(x_1^{(0)},\dots,x_{n-1}^{(0)},x_n)$ is not identically zero. Let the degree of f_i with respect to x_n be λ_i. Then the theorem is true when

* We consider $f_1,\dots,f_n$ as polynomials in x_n whose coefficients are polynomials in $x_1,\dots,x_{n-1}$. But f_i may be of degree 0 in x_n. In this case we say, for convenience, that f_i does not contain x_n.

$\lambda_1+...+\lambda_n \leqslant 1$. For, in this case, at most one of the equations (1) contains x_n. Now suppose that the theorem is true when

$$\lambda_1+...+\lambda_n \leqslant r.$$

Let us prove the theorem when $\lambda_1+...+\lambda_n = r+1$.

Without loss of generality we assume that

$$f_1 = f_{10}x_n^{\lambda_1}+...+f_{1\lambda_1}, \qquad f_2 = f_{20}x_n^{\lambda_2}+...+f_{2\lambda_2}$$

when $\lambda_1 \geqslant \lambda_2 \geqslant 1$ and f_{1i}, f_{2i} are polynomials in $x_1,..., x_{n-1}$.

Consider the system of equations

$$f_1' = f_{20}f_1 - f_{10}x_n^{\lambda_1-\lambda_2}f_2 = 0,^* \qquad f_i = 0 \quad (i = 2,..., n). \tag{3}$$

Evidently, every solution of (1) is a solution of (3). I shall prove that each non-singular solution $(x_1^{(0)},..., x_n^{(0)})$ of (1), for which $f_{20} \neq 0$, is a non-singular solution of (3).

By differentiating the first equation in (3) with respect to x_i, $(i = 1,..., n)$, we have

$$\left.\begin{aligned}\frac{\partial f_1'}{\partial x_i} &= f_{20}\frac{\partial f_1}{\partial x_i} - f_{10}x_n^{\lambda_1-\lambda_2}\frac{\partial f_2}{\partial x_i} + \frac{\partial f_{20}}{\partial x_i}f_1 - \frac{\partial f_{10}}{\partial x_i}x_n^{\lambda_1-\lambda_2}f_2 \\ &= f_{20}\frac{\partial f_1}{\partial x_i} - f_{10}x_n^{\lambda_1-\lambda_2}\frac{\partial f_2}{\partial x_i} \quad \text{at } x_\nu = x_\nu^{(0)} \quad (i = 1,..., n-1), \\ \frac{\partial f_1'}{\partial x_n} &= f_{20}\frac{\partial f_1}{\partial x_n} - f_{10}x_n^{\lambda_1-\lambda_2}\frac{\partial f_2}{\partial x_n} - (\lambda_1-\lambda_2)f_{10}x_n^{\lambda_1-\lambda_2-1}f_2 \\ &= f_{20}\frac{\partial f_1}{\partial x_n} - f_{10}x_n^{\lambda_1-\lambda_2}\frac{\partial f_2}{\partial x_n} \quad \text{at } x_\nu = x_\nu^{(0)}.\end{aligned}\right\} \tag{4}$$

Therefore

$$J(f_1', f_2,..., f_n) = f_{20}J(f_1,..., f_n) \neq 0 \quad \text{at } x_\nu = x_\nu^{(0)}. \tag{5}$$

Now suppose that $(x_1^{(0)},..., x_n^{(0)})$ is a non-singular solution of (1), for which $f_{20} = 0$. Then the solution satisfies both

$$f_1 = 0, \qquad f_2' = f_2 - f_{20}x_n^{\lambda_2} = 0, \qquad f_i = 0 \quad (i = 3,..., n), \tag{6}$$

and

$$f_1 = 0, \qquad f_{20} = 0, \qquad f_i = 0 \quad (i = 3,..., n). \tag{7}$$

By (6),

$$\frac{\partial f_2'}{\partial x_i} = \frac{\partial f_2}{\partial x_i} - \frac{\partial f_{20}}{\partial x_i}x_n^{\lambda_2} \quad (i = 1,..., n-1),$$

$$\frac{\partial f_2'}{\partial x_n} = \frac{\partial f_2}{\partial x_n} - \lambda_2 f_{20}x_n^{\lambda_2-1} = \frac{\partial f_2}{\partial x_n} \quad \text{at } x_\nu = x_\nu^{(0)}.$$

* In the case in which this equation becomes an identity, (3) has no non-singular solutions.

Hence

$$0 \neq J(f_1,\ldots,f_n) = J(f_1,f_2',f_3,\ldots,f_n) + x_n^{\lambda_2} J(f_1,f_{20},f_3,\ldots,f_n) \tag{8}$$

at $x_\nu = x_\nu^{(0)}$. This shows that $J(f_1,f_2',f_3,\ldots,f_n)$ and $J(f_1,f_{20},f_3,\ldots,f_n)$ cannot both be zero.

Therefore, by (5) and (8), each non-singular solution of (1) must be a non-singular solution of one of the systems (3), (6), (7). Since the sum of the degrees (with respect to x_n) of each of the systems is less than or equal to r, the theorem follows by induction.

I shall establish three theorems which have some interest in themselves but are not required for the treatment of the exponential sums.

THEOREM 2. *The number of solutions of* (1) *of rank r can be arranged in* $O(1)$ *sets,* each set being such that, if a certain* $n-r$ *of the variables are given, the remaining variables are determined with only* $O(1)$ *possibilities.*

Proof. For a solution of rank r, at least one of the r-rowed minors in $\mathfrak{J}(f_1,\ldots,f_m)$ is not zero. Let it be the minor in the left upper corner, say M. By Theorem 1, if $x_{r+1},\ldots,x_n$ are given, (1) has only $O(1)$ solutions for which $M \neq 0$. The theorem follows immediately.

3. Now let us consider homogeneous equations

$$g_i(x_1,\ldots,x_{n+1}) = 0 \quad (i = 1,\ldots,m), \tag{1}$$

where the $g_1,\ldots,g_m$ are homogeneous polynomials in $x_1,\ldots,x_{n+1}$. Consider the matrices

$$\mathfrak{J}(g_1,\ldots,g_m) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial g_1}{\partial x_{n+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial g_m}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial g_m}{\partial x_{n+1}} \end{bmatrix},$$

$$\mathfrak{J}_j(g_1,\ldots,g_m) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial g_1}{\partial x_{j-1}} & \frac{\partial g_1}{\partial x_{j+1}} & \cdot & \cdot & \cdot & \frac{\partial g_1}{\partial x_{n+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial g_m}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial g_m}{\partial x_{j-1}} & \frac{\partial g_m}{\partial x_{j+1}} & \cdot & \cdot & \cdot & \frac{\partial g_m}{\partial x_{n+1}} \end{bmatrix}.$$

We have the following theorems.

* The constant implied by O in this theorem and Theorem 4 depends only on m, n and the degrees of the equations considered.

THEOREM 3. *There are no solutions for which*

$$\text{Rank of } \mathfrak{J}_j < \text{Rank of } \mathfrak{J} \quad (j = 1, 2,..., n+1).$$

except the trivial solutions

$$x_1 = 0, \quad ..., \quad x_{n+1} = 0.$$

Proof. If k_i is the degree of g_i and $x_\nu = x_\nu^{(0)}$ ($\nu = 1,..., n+1$) is a solution of (1), then

$$0 = k_i g_i = x_1 \frac{\partial g_i}{\partial x_1} + ... + x_{n+1} \frac{\partial g_i}{\partial x_{n+1}} \quad \text{at } x_\nu = x_\nu^{(0)}.$$

This shows that the rank of $\mathfrak{J}$ is equal to that of $\mathfrak{J}_j$ at $x_\nu = x_\nu^{(0)}$ unless $x_j^{(0)} = 0$. The theorem follows.

THEOREM 4. *The number of non-equivalent solutions (proportional solutions being called 'equivalent') of rank r can be arranged in $O(1)$ sets, each set being such that, if a certain $n-r$ of the variables are given, the remaining variables are determined with only $O(1)$ possibilities.*

Proof. Consider first the solutions with $x_j \neq 0$. These solutions satisfy also

$$x_j^{-k_i} g_i(x_1,..., x_{n+1}) = g_i\left(\frac{x_1}{x_j},..., \frac{x_{n+1}}{x_j}\right) = 0 \quad (i = 1,..., m), \tag{1'}$$

where k_i is the degree of g_i ($1 \leqslant i \leqslant m$). Let $X_i = x_i/x_j$ or x_{i+1}/x_j according as $i \leqslant j-1$ or $i \geqslant j$. Then (1′) can be written as

$$G_i(X_1,..., X_n) = 0 \quad (i = 1,..., m). \tag{1''}$$

Plainly

$$\begin{bmatrix} \frac{\partial G_1}{\partial X_1} & \cdot & \cdot & \cdot & \frac{\partial G_1}{\partial X_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial G_m}{\partial X_1} & \cdot & \cdot & \cdot & \frac{\partial G_m}{\partial X_n} \end{bmatrix}$$

$$= x_j^{-k_i+1} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial g_1}{\partial x_{j-1}} & \frac{\partial g_1}{\partial x_{j+1}} & \cdot & \cdot & \cdot & \frac{\partial g_1}{\partial x_m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial g_m}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial g_m}{\partial x_{j-1}} & \frac{\partial g_m}{\partial x_{j+1}} & \cdot & \cdot & \cdot & \frac{\partial g_m}{\partial x_n} \end{bmatrix}$$

$$= x_j^{-k_i+1} \mathfrak{J}_j.$$

As shown in the proof of Theorem 3, for a solution with $x_j \neq 0$, the rank of $\mathfrak{J}_j$ is equal to that of $\mathfrak{J}$. Hence, corresponding to a solution $(x_1,..., x_{n+1})$ of (1) of rank r with $x_j \neq 0$, there is a solution $(X_1,..., X_n)$

of (1″) of rank r. The theorem therefore follows from Theorem 2 if we consider only non-trivial solutions. But, since there is only one trivial solution, the proof is thus completed.

4. Before going on to prove the result stated in § 1, I establish two lemmas. The constant implied by O in this section depends only on k and n.

LEMMA 1. *Let K be a finite field containing p^m elements, where p is a prime and m a positive integer. Let*

$$f(x) = a_k x^k + \ldots + a_0 \quad (a_k \neq 0,\ a_i \in k).$$

Then
$$\sum_x e^{2\pi i \mathfrak{S}[f(x)]/p} = O(p^{m(1-1/k)}), \tag{1}$$

*where $\mathfrak{S}[a]$ denotes the trace of a, and x runs over all elements of K.**

Proof. Let θ be the generating element of K with respect to the ground field π consisting of the residue classes to modulus p. Then every element x of K can be written as

$$x = \alpha_0 + \alpha_1 \theta + \ldots + \alpha_{m-1} \theta^{m-1} \quad (\alpha_i \in \pi).$$

Since $\mathfrak{S}[x+y] = \mathfrak{S}[x] + \mathfrak{S}[y]$, we have, for $a \in K$,

$$\sum_x e^{2\pi i \mathfrak{S}[ax]/p} = \sum_{\alpha_0=1}^{p} \ldots \sum_{\alpha_{m-1}=1}^{p} e^{2\pi i(\alpha_0 \mathfrak{S}[a] + \ldots + \alpha_{m-1} \mathfrak{S}[a\theta^{m-1}])/p}$$
$$= \begin{cases} p^m & \text{if } \mathfrak{S}[a\theta^k] = 0 \quad \text{for } k = 0,\ldots,m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Further, $\mathfrak{S}[a\theta^k] = 0$ for $k = 0,\ldots,m-1$ implies $a = 0$. In fact, we have $\mathfrak{S}[ax] = 0$ for all $x \in K$. If $a \neq 0$, we should have

$$0 = \mathfrak{S}[aa^{-1}] = 1,$$

which is absurd. Hence†

$$\sum_{b_1} \ldots \sum_{b_k} \left| \sum_x e^{2\pi i \mathfrak{S}[b_1 x + \ldots + b_k x^k]/p} \right|^{2k}$$
$$= \sum_{x_1} \ldots \sum_{x_k} \sum_{y_1} \ldots \sum_{y_k} \sum_{b_1} \ldots \sum_{b_k} e^{2\pi i \mathfrak{S}[b_1(x_1+\ldots+x_k-y_1-\ldots-y_k)+\ldots+b_k(x_1^k+\ldots-y_k^k)]/p}$$
$$= \sum_{\substack{x_1 \ldots x_k\, y_1 \ldots y_k \\ x_1+\ldots+x_k = y_1+\ldots+y_k \\ \cdots\cdots \\ x_1^k+\ldots+x_k^k = y_1^k+\ldots+y_k^k}} p^{mk} = O(p^{2mk}). \tag{2}$$

* Setting $m = 1$ and $\mathfrak{S}[x] = x$ we get Mordell's theorem. In the proof we can simply put $\theta = 0$ and omit all unnecessary words.

† We consider $a_1,\ldots, a_k$ as fixed and $b_1,\ldots, b_k$ as variables that assume all values in K.

If λ and μ are both in K and $\lambda = 0$, then

$$\sum_x e^{2\pi i \mathfrak{S}[f(\lambda x+\mu)]/p} = \sum_x e^{2\pi i \mathfrak{S}[f(x)]/p}.$$

The transformation $x = \lambda x' + \mu$ ($\lambda \in K$, $\mu \in K$, $\lambda \neq 0$) transforms the sum in (1) into at least $p^m(p^m-1)/k$ different terms in the sum (2). In fact, if $f(\lambda x+\mu) \equiv f(x)$,* then

$$\lambda^k a_k = a_k, \qquad n\lambda^{k-1}\mu a_k + \lambda^{k-1}a_{k-1} = a_{k-1}.$$

The first equation gives at most k values of λ and the second equation determines μ uniquely. Hence

$$\frac{p^m(p^m-1)}{k}\left|\sum_x e^{2\pi i \mathfrak{S}[f(x)]/p}\right|^{2k} = O(p^{2mk}),$$

and
$$\sum_x e^{2\pi i \mathfrak{S}[f(x)]/p} = O(p^{m(1-1/k)}).$$

LEMMA 2. *Let $f(x_1,\dots,x_n)$ be a polynomial not zero in K, then the number of solutions of*

$$f(x_1,\dots,x_n) = 0 \tag{3}$$

is $O(p^{m(n-1)})$.

Proof. The proof is evident for $n = 1$. Suppose that the lemma is true for $n-1$ unknowns. Let $f(x_1,\dots,x_n) = g_0 x_n^k + \dots + g_k$ where g_i are polynomials in $x_1,\dots,x_{n-1}$ and $g_0 \not\equiv 0$. The number of solutions of $f = 0$, $g_0 \neq 0$ is evidently $O(p^{m(n-1)})$ and that of $f = 0$, $g_0 = 0$ is, by our supposition, $O(p^{m(n-2)}p^m) = O(p^{m(n-1)})$. Therefore the number of solutions of (3) is $O(p^{m(n-1)})$.

Now we can prove

THEOREM 5. *Let $f(x_1,\dots,x_n)$ be a polynomial of degree $k \geqslant 2n$ in the field K containing p^m elements. Let*

$$f(\alpha_1 t+\beta_1,\dots,\alpha_n t+\beta_n) = \sum_{r=0}^{k} F_r(\alpha_1,\dots,\alpha_n;\, \beta_1,\dots,\beta_n)t^r.$$

Suppose the matrix

$$\mathfrak{M} = \begin{bmatrix} \dfrac{\partial F_1}{\partial \alpha_1} & \cdot & \cdot & \cdot & \dfrac{\partial F_1}{\partial \beta_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial F_k}{\partial \alpha_1} & \cdot & \cdot & \cdot & \dfrac{\partial F_k}{\partial \beta_n} \end{bmatrix}$$

is of rank $2n$, i.e. a certain $2n \times 2n$ minor is not identically zero. Then

$$\sum_{x_1} \dots \sum_{x_n} e^{2\pi i \mathfrak{S}[f(x_1,\dots,x_n)]/p} = O(p^{mn(1-1/k)}).$$

* This means that they are the same polynomial in K.

Proof. Let $J(\alpha_1,\ldots,\beta_n)$ be a minor which is not identically zero. We regard J as a polynomial in β_n and denote its leading coefficient* by $J'(\alpha_1,\ldots,\beta_{n-1})$. Then regard J' as a polynomial in β_{n-1} and denote its leading coefficient by $J''(\alpha_1,\ldots,\beta_{n-2})$ and so on. Finally, we obtain a polynomial $J_0(\alpha_1,\ldots,\beta_1)$ such that, if $J_0(\alpha_1^{(0)},\ldots,\beta_1^{(0)}) \neq 0$, the polynomial $J(\alpha_1^{(0)},\ldots,\beta_1^{(0)},\beta_2,\ldots,\beta_n)$ is not identically zero.

If $\alpha_1 \neq 0$,

$$\sum_x \sum_{\beta_2} \cdots \sum_{\beta_n} e^{2\pi i \mathfrak{S}[f(\alpha_1 x+\beta_1,\ldots,\alpha_n x+\beta_n)]/p} = \sum_{x_1} \cdots \sum_{x_n} e^{2\pi i \mathfrak{S}[f(x_1,\ldots,x_n)]/p}.$$

Since F_k depends only on $\alpha_1,\ldots,\alpha_n$, the number of solutions of $\alpha_1 J_0 F_k = 0$ is $O(p^{mn})$ by Lemma 2, and hence the number of sets $(\alpha_1,\ldots,\alpha_n,\beta_1)$ for which $\alpha_1 J_0 F_k \neq 0$ is $p^{(n+1)m} - O(p^{mn})$. It follows that

$$\{p^{(n+1)m} - O(p^{mn})\} \left| \sum_{x_1} \cdots \sum_{x_n} e^{2\pi i \mathfrak{S}[f(x_1,\ldots,x_n)]/p} \right|^{2k}$$
$$= \sum_{\substack{\alpha_1 \\ \alpha_1 J_0 F_k \neq 0}} \cdots \sum_{\alpha_n} \sum_{\beta_1} \left| \sum_x \sum_{\beta_2} \cdots \sum_{\beta_n} e^{2\pi i \mathfrak{S}[f(\alpha_1 x+\beta_1,\ldots,\alpha_n x+\beta_n)]/p} \right|^{2k}$$
$$= \sum_{\substack{\alpha_1 \\ \alpha_1 J_0 F_k \neq 0}} \cdots \sum_{\alpha_n} \sum_{\beta_1} \left| \sum_x \sum_{\substack{\beta_2 \\ J\neq 0}} \cdots \sum_{\beta_n} e^{2\pi i \mathfrak{S}[f(\alpha_1 x+\beta_1,\ldots,\alpha_n x+\beta_n)]/p} + \right.$$
$$\left. + \sum_x \sum_{\substack{\beta_2 \\ J=0}} \cdots \sum_{\beta_n} e^{2\pi i \mathfrak{S}[f(\alpha_1 x+\beta_1,\ldots,\alpha_n x+\beta_n)]/p} \right|^{2k}.$$

By Hölder's inequality, this does not exceed

$$2^{2k-1} \sum_{\substack{\alpha_1 \\ \alpha_1 J_0 F_k \neq 0}} \cdots \sum_{\alpha_n} \sum_{\beta_1} \left| \sum_x \sum_{\substack{\beta_2 \\ J\neq 0}} \cdots \sum_{\beta_n} e^{2\pi i \mathfrak{S}[f(\alpha_1 x+\beta_1,\ldots,\alpha_n x+\beta_n)]/p} \right|^{2k} +$$
$$+ 2^{2k-1} \sum_{\substack{\alpha_1 \\ \alpha_1 J_0 F_k \neq 0}} \cdots \sum_{\alpha_n} \sum_{\beta_1} \left| \sum_x \sum_{\substack{\beta_2 \\ J=0}} \cdots \sum_{\beta_n} e^{2\pi i \mathfrak{S}[f(\alpha_1 x+\beta_1,\ldots,\alpha_n x+\beta_n)]/p} \right|^{2k}$$
$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

By Lemmas 1 and 2,

$$\Sigma_2 = O\Big(\sum_{\substack{\alpha_1 \\ J_0 \neq 0}} \cdots \sum_{\alpha_n} \sum_{\beta_1} \Big| \sum_{\substack{\beta_2 \\ J=0}} \cdots \sum_{\beta_n} p^{m(1-1/k)} \Big|^{2k} \Big)$$
$$= O(p^{m[(n+1)+2k(n-2+1-1/k)]}) = O(p^{m(2kn-2k+n-1)}). \tag{4}$$

By Hölder's inequality,

$$\Sigma_1 \leqslant 2^{2k-1} p^{m(n-1)(2k-1)} \sum_{\substack{\alpha_1 \\ \alpha_1 J_0 F_k J \neq 0}} \cdots \sum_{\alpha_n} \sum_{\beta_1} \cdots \sum_{\beta_n} \left| \sum_x e^{2\pi i \mathfrak{S}[f(\alpha_1 x+\beta_1,\ldots,\alpha_n x+\beta_n)]/p} \right|^{2k}.$$

* That is, the coefficient of the highest power of β_n.

Suppose that

$$f(\alpha_1 x+\beta_1,\ldots,\alpha_n x+\beta_n) = F_k x^k+\ldots+F_1 x+F_0,$$

and that J is the Jacobian of $F_{r_1},\ldots,F_{r_{2n}}$ $(1 \leqslant r_1 < \ldots < r_{2n} \leqslant k)$. Then the number of solutions of the system of equations

$$F_{r_i} \equiv F_{r_i}(\alpha_1,\ldots,\beta_n) = a_{r_i} \qquad (i = 1,\ldots, 2n)$$

in the $2n$ unknowns $\alpha_1,\ldots,\beta_n$ for which $J \neq 0$ is $O(1)$, by Theorem 1. It follows that

$$\Sigma_1 = p^{m(n-1)(2k-1)}O\Big(\sum_{a_k}\cdots\sum_{a_1}\Big|\sum_x e^{2\pi i \mathfrak{S}[a_k x^k+\ldots+a_1 x]/p}\Big|^{2k}\Big).$$

Hence, by the inequality from which we deduced Lemma 1,

$$\Sigma_1 = O(p^{m(n-1)(2k-1)}p^{2km}) = O(p^{m(2kn-n+1)}). \tag{5}$$

By (4) and (5)

$$\sum_{x_1}\cdots\sum_{x_n} e^{2\pi i \mathfrak{S}[f(x_1,\ldots,x_n)]/p} = O(p^{m[(2kn-n+1)-(n+1)]/2k}) = O(p^{mn(1-1/k)}),$$

the result required.

I wish to express my hearty thanks to the referees for their kind suggestions.

CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$

By R. P. BAMBAH (*Delhi*), S. CHOWLA (*Lahore*), H. GUPTA (*Hoshiarpur*), and D. B. LAHIRI (*Calcutta*)

[Received 28 August 1946]

1. FOLLOWING Ramanujan, we define $\tau(n)$ by the relation

$$\sum_1^\infty \tau(n)x^n = x\prod_1^\infty (1-x^n)^{24},$$

where $|x| < 1$.

It is known that (1)

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}; \tag{a}$$

$$\tau(n) \equiv n\sigma(n) \pmod{5}, \tag{b}$$

where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n and

$$\sigma(n) = \sigma_1(n);$$

$$\tau(n) \equiv 0 \pmod{23} \tag{c}$$

if $\quad n \equiv k \pmod{23} \quad$ and $\quad \left(\frac{k}{23}\right) = -1;$

$$\tau(n) \equiv 0 \pmod{7} \tag{d}$$

if $\quad n \equiv 0,\ 3,\ 5,\ 6 \pmod{7};$

$$\sigma(n) \equiv 1 \pmod{2} \tag{e}$$

if and only if n is an odd square;

$$\tau(jn-1) \equiv \sigma(jn-1) \pmod{j} \tag{f}$$

if j is a divisor of 24.

Recently Lahiri has shown (2) how to determine the residues of $\tau(n)$ modulo 3. His results are equivalent to the simple formulae

$$\tau(n) \equiv \sigma(n) \pmod{3} \quad \text{if } (n, 3) = 1; \tag{A}$$

$$\tau(3n) \equiv 0 \pmod{3}. \tag{B}$$

After reading an advance copy of Lahiri's paper, we were able to get new and simple proofs of the striking result (A), which, curiously enough, seems to have been overlooked by previous authors.

2. Proof of (A)

As in (3), we have $$1728 \sum_1^\infty \tau(n)x^n = Q^3 - R^2, \tag{1}$$

where $$Q = 1+240 \sum_1^\infty \sigma_3(n)x^n \tag{2}$$

and $$R = 1-504 \sum_1^\infty \sigma_5(n)x^n. \tag{3}$$

Now, if $\phi(x)$ be any power series with integral coefficients, then

$$\{\phi(x)\}^p - \phi(x^p) = \sum_1^\infty e_n x^n,$$

where $e_n \equiv 0 \pmod p$, p being a prime.

In particular, when $p = 3$, we see that

$$\left\{\sum_1^\infty \sigma_3(n)x^n\right\}^3 = \sum_1^\infty c_n x^n,$$

where $c_n \equiv 0 \pmod 3$ if $(n,3) = 1$;

i.e. $$\sum_{\substack{u+v+w=n\\ u,v,w\geqslant 1}} \sigma_3(u)\sigma_3(v)\sigma_3(w) \equiv 0 \pmod 3 \quad \text{if } (n,3)=1. \tag{4}$$

From (2), $$Q^3 = \sum_0^\infty d_n x^n,$$

where, for $n \geqslant 1$,

$$d_n = 720\sigma_3(n) + 3.240^2 \sum_{\substack{u+v=n\\ u,v\geqslant 1}} \sigma_3(u)\sigma_3(v) + 240^3 \sum_{\substack{u+v+w=n\\ u,v,w\geqslant 1}} \sigma_3(u)\sigma_3(v)\sigma_3(w). \tag{5}$$

Again, by (3), we have

$$\sum_{\substack{u+v=n\\ u,v\geqslant 1}} \sigma_3(u)\sigma_3(v) = \tfrac{1}{120}\{\sigma_7(n)-\sigma_3(n)\}, \tag{6}$$

so that, if $(n,3) = 1$, we have from (4), (5), (6)

$$\begin{aligned} d_n &\equiv 720\sigma_3(n) + 1440\{\sigma_7(n)-\sigma(n)\} \pmod{81} \\ &\equiv 9\{\sigma_3(n) - 2\sigma_7(n)\} \pmod{81}. \end{aligned} \tag{7}$$

Moreover, from (3), $$R^2 = \sum_0^\infty l_n x^n,$$

where, for $n \geqslant 1$,

$$\begin{aligned} l_n &= -1008\sigma_5(n) + 504^2 \sum_{\substack{u+v=n\\ u,v\geqslant 1}} \sigma_5(u)\sigma_5(v) \\ &\equiv -36\sigma_5(n) \pmod{81}. \end{aligned} \tag{8}$$

Hence, from (1), (7), (8), if $(n,3) = 1$, then

$$1728\tau(n) \equiv 9\{\sigma_3(n) - 2\sigma_7(n) + 4\sigma_5(n)\} \pmod{81},$$

i.e., if $(n,3) = 1$,

$$3\tau(n) \equiv \sigma_3(n) - 2\sigma_7(n) + 4\sigma_5(n) \pmod 9, \tag{9}$$

i.e. $$3\{\tau(n)-\sigma(n)\} \equiv \sum_{d|n} \{d^3 - 2d^7 + 4d^5 - 3d\} \pmod 9.$$

Now

$$d^3-2d^7+4d^5-3d = d(d^2-1)(3+2d^2-2d^4) = 3d(d^2-1)-2d^3(d^2-1)^2$$
$$\equiv 0 \pmod 9. \tag{10}$$

Hence, from (9) and (10), if $(n,3) = 1$,

$$\tau(n) \equiv \sigma(n) \pmod 3.$$

3. Proceeding on the same lines, we now prove

$$\tau(n) \equiv \sigma(n) \pmod 8 \tag{C}$$

if $(n,2) = 1$.

Since $$d^5 \equiv d^3 \pmod 8,$$

we have $$\sum_{\substack{u+v=n\\ u,v\geqslant 1}} \sigma_5(u)\sigma_5(v) \equiv \sum_{\substack{u+v=n\\ u,v\geqslant 1}} \sigma_3(u)\sigma_3(v) \pmod 8$$

$$\equiv \tfrac{1}{120}\{\sigma_7(n)-\sigma_3(n)\} \pmod 8. \tag{11}$$

From (5) and (6) for $n \geqslant 1$, we have

$$d_n \equiv 720\sigma_3(n)+1440\{\sigma_7(n)-\sigma_3(n)\} \pmod{2^9}$$
$$\equiv 208\{2\sigma_7(n)-\sigma_3(n)\} \pmod{2^9}. \tag{12}$$

From (8) and (11) for $n \geqslant 1$,

$$l_n \equiv -1008\sigma_5(n)+\frac{504^2}{120}\{\sigma_7(n)-\sigma_3(n)\} \pmod{2^9}$$

$$\equiv 16\sigma_5(n)+\frac{8}{15}\{\sigma_7(n)-\sigma_3(n)\} \pmod{2^9}. \tag{13}$$

Now from (1), (12), (13), for $n \geqslant 1$, we have

$$15.1728\tau(n) \equiv 15.208\{2\sigma_7(n)-\sigma_3(n)\}-15.16\sigma_5(n)-8\{\sigma_7(n)-\sigma_3(n)\} \pmod{2^9},$$

i.e. $$15.216\tau(n) \equiv 11\sigma_7(n)-30\sigma_5(n)-5\sigma_3(n) \pmod{2^6},$$

i.e. $$24\{\sigma(n)-\tau(n)\} \equiv \sum_{d|n}\{11d^7-30d^5-5d^3+24d\} \pmod{2^6}.$$

Now, if $(n,2) = 1$, every divisor d of n is odd and

$$11d^7-30d^5-5d^3+24d = d(11d^2-8)(d^2-1)^2-32(d^2-1)$$
$$\equiv 0 \pmod{2^6}. \tag{14}$$

Hence, if $(n,2) = 1$,

$$\tau(n) \equiv \sigma(n) \pmod 8.$$

4. From §§ 2, 3 we have

If j be a divisor of 24 *and* $(n,j) = 1$, *then*

$$\tau(n) \equiv \sigma(n) \pmod{j}. \tag{D}$$

It may be noted that (e) and (f) are particular cases of (D).

REFERENCES

1. Results (a)–(d) are due to Ramanujan, see Hardy's *Ramanujan* (Cambridge, 1940), 165–9; (e) is due to Gupta and (f) to Ramanathan. See Gupta, *Proc. Benares Math. Soc.* (1943), and *J. of Indian Math. Soc.* 9 (1945), 59–60; Ramanathan, *J. of Indian Math. Soc.* 9 (1945), 55–9. In the papers of Gupta cited, two of our results, viz. (A) and (C), are proved by other methods.
2. In an unpublished paper.
3. Ramanujan, *Collected Papers* (Cambridge, 1927), 142–6.

ON THE DETERMINATION OF THE JUMP OF A FUNCTION BY ITS FOURIER COEFFICIENTS

By M. L. MISRA (*Udaipur*)

[Received 19 October 1946]

1. LET $f(x)$ be integrable in the sense of Lebesgue over the interval $(-\pi,\pi)$ and be periodic outside with a period 2π. Let the Fourier series of $f(x)$ be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{1.1}$$

Then the conjugate series of (1.1) is

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx). \tag{1.2}$$

Suppose that

$$\phi(t)\equiv\tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},\quad s\equiv s(x),$$

$$\psi(t)\equiv f(x+t)-f(x-t)-D,\qquad D\equiv D(x),$$

$$\psi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int\limits_t^\pi\left(\log\frac{u}{t}\right)^{\alpha-1}\frac{\psi(u)}{u}\,du\quad(\alpha>0),$$

$$\psi_0(t)=\psi(t).$$

It is known* that

$$\psi_\beta(t)=\frac{1}{\Gamma(\beta-\alpha)}\int\limits_t^\pi\left(\log\frac{u}{t}\right)^{\beta-\alpha-1}\frac{\psi_\alpha(u)}{u}\,du\quad(\beta>\alpha). \tag{1.3}$$

Szász† gave the following theorem for the determination of the jump $f(x+0)-f(x-0)$ or generalized jump $D(x)$ of a function $f(x)$ by its Fourier coefficients:

THEOREM 1. *If there exists a number $D\equiv D(x)$ such that*

$$\int\limits_0^t\psi(t)\,dt=o(t)\quad\textit{and}\quad\int\limits_0^t|\psi(t)|\,dt=O(t),$$

as $t\to 0$, then $$\lim_{n\to\infty}\{\bar{S}^1_{2n}(x)-\bar{S}^1_n(x)\}=\frac{\log 2}{\pi}D(x),$$

where $\bar{S}^1_n(x)$ is the sequence of the arithmetic means of the partial sums of the conjugate series.

* Wang (6). † Szász (5).

Theorem 1 was generalized by Chow* in the form of

THEOREM 2. *Under the same hypothesis as in Szász's theorem*

$$\lim_{n\to\infty}\{\bar{S}^{\alpha}_{2n}(x)-\bar{S}^{\alpha}_{n}(x)\} = \frac{\log 2}{\pi} D(x),$$

for $\alpha > 0$, *where* $\bar{S}^{\alpha}_{n}(x)$ *is the n-th Cesàro mean of order* α *of the conjugate series.*

Further generalizations of this theorem have been given by Minakshisundaram.†

Definition. A series $\sum c_n$ is said to be summable by Riesz's logarithmic mean of order $k > 0$, or summable $(R, \log n, k)$ to the sum s, provided that

$$R_k(\omega) = \frac{1}{(\log\omega)^k} \sum_{n<\omega} \left(\log\frac{\omega}{n}\right)^k c_n$$

tends to a limit s as $\omega \to \infty$.

The object of this paper is to prove the theorems:

THEOREM A. *If*

$$\int_t^{\pi} \frac{|\psi(t)|}{t}\,dt = o\left(\log\frac{1}{t}\right), \tag{1.4}$$

as $t \to 0$, *then* $\lim_{\omega\to\infty}[\bar{R}_1(2\omega)-\bar{R}_1(\omega)] = \frac{\log 2}{2\pi} D(x),$

where $\bar{R}_1(\omega)$ *is Riesz's logarithmic mean of order unity of the conjugate series.*

THEOREM B. *If*

$$\int_t^{\pi} \frac{|\psi_\beta(t)|}{t}\,dt = o\left\{\left(\log\frac{1}{t}\right)^{\beta+1}\right\}, \tag{1.5}$$

as $t \to 0$, *then*

$$\lim_{\omega\to\infty}[\bar{R}_{\beta+1}(\lambda\omega)-\bar{R}_{\beta+1}(\omega)] = \frac{D}{(\beta+2)\pi}\log\lambda \quad (\lambda > 1;\ \beta \geqslant 0),$$

where $\bar{R}_{\beta+1}(\omega)$ *is Riesz's logarithmic mean of order* $(\beta+1)$ *of the conjugate series.*

Theorem A is the case $\lambda = 2$, $\beta = 0$ of Theorem B.

I also prove the following generalization of the well-known

* Chow (**1**). † Minakshisundaram (**4**).

theorem* on the order of magnitude of the partial sums $s_n(x)$, $\bar{s}_n(x)$ of the Fourier series of $f(x)$ and its conjugate series respectively:

THEOREM C. *(a) If*

$$\int_t^\pi \frac{|\phi(t)|}{t}\,dt = o\left(\log\frac{1}{t}\right),$$

as $t \to 0$, *then* $$s_n(x) = o(\log n).$$

(b) If $$\int_t^\pi \frac{|\psi(t)|}{t}\,dt = o\left(\log\frac{1}{t}\right),$$

as $t \to 0$, *then* $$\bar{s}_n(x) \sim \frac{D}{\pi}\log n.$$

It is easily seen that, if

$$\int_0^t |\phi(t)|\,dt = o(t),$$

as $t \to 0$, then $$\int_t^\pi \frac{|\phi(t)|}{t}\,dt = o\left(\log\frac{1}{t}\right).$$

On the other hand, if

$$\int_t^\pi \frac{|\phi(t)|}{t}\,dt = o\left(\log\frac{1}{t}\right),$$

then $$\int_0^t |\phi(t)|\,dt = o\left(t\log\frac{1}{t}\right).$$

Hence Theorem C generalizes the theorem on the orders of magnitude referred to above which holds under the conditions

$$\int_0^t |\phi(t)|\,dt = o(t), \qquad \int_0^t |\psi(t)|\,dt = o(t).$$

2. I shall make use of functions $S_k(t)$ given by

$$S_k(t) = \int_0^1 \left(\log\frac{1}{u}\right)^k \sin tu\,du \quad (k > -1).$$

* Hardy and Rogosinski (**3**) (49, Theorem 64).

It is known* that

$$\frac{d}{dt}[tS_k(t)] = kS_{k-1}(t) \quad (k > 0), \tag{2.1}$$

$$S_{r+s+1}(t) = \frac{\Gamma(r+s+2)}{\Gamma(r+1)\Gamma(s+1)} \int_0^1 S_s(ut)\left(\log\frac{1}{u}\right)^r du \quad (r, s > -1). \tag{2.2}$$

We shall require the lemma

LEMMA. *If $k \geqslant 1$, then for all $\omega > 1$,*

$$R_k'(\omega) = \frac{k}{\omega \log \omega}[R_{k-1}(\omega) - R_k(\omega)].$$

We have

$$R_k'(\omega) = \frac{d}{d\omega}\left[\sum_{n<\omega}\left(1-\frac{\log n}{\log \omega}\right)^k c_n\right]$$

$$= \frac{k}{\omega(\log\omega)^2}\sum_{n<\omega}\left(1-\frac{\log n}{\log\omega}\right)^{k-1}(\log n)c_n$$

$$= \frac{k}{\omega\log\omega}\sum_{n<\omega}\left\{\left(1-\frac{\log n}{\log\omega}\right)^{k-1} - \left(1-\frac{\log n}{\log\omega}\right)^k\right\}c_n,$$

which proves the lemma.

3. Proof of Theorem C

(*a*) We have

$$s_n - s = \frac{1}{\pi}\int_0^\pi \phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right\}\phi(t)\frac{\sin nt}{t}\,dt + o(1)$$

$$= \frac{2}{\pi}(I_1 + I_2) + o(1), \quad \text{say.}$$

Now

$$|I_1| \leqslant n\int_0^{\pi/n}|\phi(t)|\,dt$$

$$= n\left[-t\int_t^\pi \frac{|\phi(t)|}{t}\,dt\right]_0^{\pi/n} + n\int_0^{\pi/n}\left\{\int_t^\pi \frac{|\phi(t)|}{t}\,dt\right\}dt$$

$$= o(\log n) \quad (n \to \infty),$$

* Wang (6).

if $\int\limits_{t}^{\pi} \frac{|\phi(t)|}{t}\,dt = o\left(\log\frac{1}{t}\right)$ as $t \to 0$. Also

$$I_2 = O\left(\int\limits_{\pi/n}^{\pi} \frac{|\phi(t)|}{t}\,dt\right) = o(\log n).$$

Hence

$$s_n(x) = s+o(\log n)+o(1)$$
$$= o(\log n) \quad (n \to \infty),$$

which proves the part (a) of Theorem C.

(b) We have

$$\bar{s}_n(x) = \frac{1}{2\pi}\int\limits_0^{\pi} \{f(x+t)-f(x-t)\}\cot\tfrac{1}{2}t(1-\cos nt)\,dt+o(1)$$
$$= \frac{D}{2\pi}\int\limits_0^{\pi} \cot\tfrac{1}{2}t(1-\cos nt)\,dt+$$
$$+\frac{1}{2\pi}\left\{\int\limits_0^{\pi/n}+\int\limits_{\pi/n}^{\pi}\right\}\psi(t)\cot\tfrac{1}{2}t(1-\cos nt)\,dt+o(1)$$
$$= J_1+J_2+J_3+o(1), \quad \text{say.}$$

Now*

$$J_1 = \frac{D}{\pi}[\log 2n+\gamma]+o(1),$$

where γ is Euler's constant. Also

$$|J_2| \leqslant n\int\limits_0^{\pi/n} |\psi(t)|\,dt$$
$$= o(\log n) \quad (n \to \infty),$$

if $\int\limits_{t}^{\pi} \frac{|\psi(t)|}{t}\,dt = o\left(\log\frac{1}{t}\right)$, as $t \to 0$.

Again

$$J_3 = O\left(\int\limits_{\pi/n}^{\pi} \frac{|\psi(t)|}{t}\,dt\right) = o(\log n).$$

Hence

$$\bar{s}_n(x) = \frac{D}{\pi}\log n+O(1)+o(\log n)$$
$$\sim \frac{D}{\pi}\log n,$$

which proves the part (b) of the Theorem C.

* Hardy and Rogosinski (3), 47.

4. Proof of Theorem A

We have, if

$$\psi_1(t) = \int_t^\pi \frac{\psi(t)}{t}\,dt,$$

$$\begin{aligned}
\bar{R}_0(\omega) &= \sum_{n<\omega}(b_n\cos nx - a_n\sin nx) = \sum_{n<\omega} B_n \\
&= \frac{1}{2\pi}\int_0^\pi \{f(x+t)-f(x-t)\}\cot\tfrac{1}{2}t(1-\cos\omega t)\,dt + o(1) \\
&= \frac{D}{2\pi}\int_0^\pi \cot\tfrac{1}{2}t(1-\cos\omega t)\,dt + \frac{1}{2\pi}\int_0^\pi \psi(t)\cot\tfrac{1}{2}t(1-\cos\omega t)\,dt + o(1) \\
&= \frac{D}{\pi}(\log 2\omega+\gamma) + \frac{1}{2\pi}\int_0^\pi \psi(t)\left(\cot\tfrac{1}{2}t - \frac{2}{t}\right)(1-\cos\omega t)\,dt + \\
&\qquad + \frac{1}{\pi}\int_0^\pi \psi(t)\,\frac{1-\cos\omega t}{t}\,dt + o(1) \\
&= \frac{D}{\pi}\log\omega + \frac{1}{\pi}\int_0^\pi \psi(t)\,\frac{1-\cos\omega t}{t}\,dt + O(1) \\
&= \frac{D}{\pi}\log\omega + \frac{1}{\pi}\Big[-(1-\cos\omega t)\psi_1(t)\Big]_0^\pi + \frac{\omega}{\pi}\int_0^\pi \psi_1(t)\sin\omega t\,dt + O(1) \\
&= \frac{D}{\pi}\log\omega + \frac{\omega}{\pi}\int_0^\pi \psi_1(t)\sin\omega t\,dt + O(1) \quad (\omega\to\infty);
\end{aligned}$$

$$\begin{aligned}
\bar{R}_1(\omega) &= \frac{1}{\log\omega}\sum_{n<\omega}\left(\log\frac{\omega}{n}\right)B_n = \frac{1}{\log\omega}\int_1^\omega \bar{R}_0(x)\,\frac{dx}{x} \\
&= \frac{D}{\pi\log\omega}\int_1^\omega \frac{\log x}{x}\,dx + \frac{1}{\pi\log\omega}\int_1^\omega dx\int_0^\pi \psi_1(t)\sin xt\,dt + O(1) \\
&= \frac{D}{2\pi}\log\omega + \frac{1}{\pi\log\omega}\int_0^\pi \psi_1(t)\,dt\int_1^\omega \sin xt\,dx + O(1) \\
&= \frac{D}{2\pi}\log\omega + \frac{1}{\pi\log\omega}\int_0^\pi \psi_1(t)\,\frac{1-\cos\omega t}{t}\,dt + O(1),
\end{aligned}$$

since
$$\int_0^\pi \psi_1(t)\,\frac{1-\cos t}{t}\,dt = O(1),$$
$\psi_1(t)$ being integrable (L) in $(0,\pi)$.

Hence, by the lemma,

$$\pi\frac{d}{d\omega}\bar{R}_1(\omega) = \frac{\pi}{\omega\log\omega}[\bar{R}_0(\omega)-\bar{R}_1(\omega)]$$
$$= \frac{\pi}{\omega\log\omega}\Bigg[\frac{D}{2\pi}\log\omega+\frac{1}{\pi}\int_0^\pi \psi(t)\,\frac{1-\cos\omega t}{t}\,dt -$$
$$-\frac{1}{\pi\log\omega}\int_0^\pi \psi_1(t)\,\frac{1-\cos\omega t}{t}\,dt+O(1)\Bigg]$$
$$= \frac{D}{2\omega}+\frac{1}{\omega\log\omega}I-\frac{1}{\omega(\log\omega)^2}J+o\left(\frac{1}{\omega}\right).$$

Now
$$I = o(\log\omega),$$
as in the proof of Theorem C. Also
$$|J| \leqslant \left|\int_0^{\pi/\omega}\psi_1(t)\,\frac{1-\cos\omega t}{t}\,dt\right|+\left|\int_{\pi/\omega}^{\pi}\psi_1(t)\,\frac{1-\cos\omega t}{t}\,dt\right|$$
$$= \omega\int_0^{\pi/\omega} o\left(\log\frac{1}{t}\right)dt+\int_{\pi/\omega}^{\pi}\frac{1}{t}\,o\left(\log\frac{1}{t}\right)dt$$
$$= o(\log\omega)+o\{(\log\omega)^2\} = o\{(\log\omega)^2\}.$$

Hence
$$\frac{d}{d\omega}\bar{R}_1(\omega) = \frac{D}{2\pi\omega}+o\left(\frac{1}{\omega}\right).$$

Now
$$\bar{R}_1(2\omega)-\bar{R}_1(\omega) = \int_\omega^{2\omega}\frac{d}{dt}\bar{R}_1(t)\,dt$$
$$= \frac{D}{2\pi}\int_\omega^{2\omega}\frac{dt}{t}+\int_\omega^{2\omega}o\left(\frac{1}{t}\right)dt$$
$$= \frac{D}{2\pi}\log 2+o(1)\quad(\omega\to\infty),$$

which proves Theorem A.

5. Proof of Theorem B

We have

$$\bar{R}_0(\omega) = \sum_{n<\omega} B_n$$

$$= \frac{D}{\pi}\log\omega + \frac{\omega}{\pi}\int_0^\pi \psi_1(t)\sin\omega t\,dt + O(1).$$

Also, for $k > 0$,

$$\bar{R}_k(\omega) = \frac{1}{(\log\omega)^k}\sum_{n<\omega}\left(\log\frac{\omega}{n}\right)^k B_n = \frac{k}{(\log\omega)^k}\int_1^\omega \left(\log\frac{\omega}{x}\right)^{k-1}\frac{\bar{R}_0(x)}{x}\,dx$$

$$= \frac{D}{\pi}\frac{k}{(\log\omega)^k}\int_1^\omega \left(\log\frac{\omega}{x}\right)^{k-1}\frac{\log x}{x}\,dx +$$

$$+ \frac{k}{\pi(\log\omega)^k}\int_1^\omega \left(\log\frac{\omega}{x}\right)^{k-1}dx\int_0^\pi \psi_1(t)\sin xt\,dt + O(1)$$

$$= \frac{D\log\omega}{(k+1)\pi} + \frac{k}{\pi(\log\omega)^k}\int_0^\pi \psi_1(t)\,dt\int_1^\omega \left(\log\frac{\omega}{x}\right)^{k-1}\sin xt\,dx + O(1)$$

$$= \frac{D\log\omega}{(k+1)\pi} + \frac{k}{\pi(\log\omega)^k}\int_0^\pi \psi_1(t)\,dt\int_0^\omega \left(\log\frac{\omega}{x}\right)^{k-1}\sin xt\,dx + O(1)$$

$$= \frac{D\log\omega}{(k+1)\pi} + \frac{k\omega}{\pi(\log\omega)^k}\int_0^\pi \psi_1(t)S_{k-1}(\omega t)\,dt + O(1).$$

Hence, by the lemma, we have

$$\bar{R}'_{\beta+1}(\omega) = \frac{(\beta+1)}{\omega\log\omega}[\bar{R}_\beta(\omega) - \bar{R}_{\beta+1}(\omega)]$$

$$= \frac{D}{(\beta+2)\pi\omega} + \frac{\beta(\beta+1)}{\pi(\log\omega)^{\beta+1}}\int_0^\pi \psi_1(t)S_{\beta-1}(\omega t)\,dt -$$

$$- \frac{(\beta+1)^2}{\pi(\log\omega)^{\beta+2}}\int_0^\pi \psi_1(t)S_\beta(\omega t)\,dt + O\left(\frac{1}{\omega\log\omega}\right)$$

$$= \frac{D}{(\beta+2)\pi\omega} + R_1 - R_2 + o\left(\frac{1}{\omega}\right), \quad \text{say.}$$

Now, integrating by parts h times where $h = [\beta]$, if β is not an integer, we have

$$\frac{\pi(\log\omega)^{\beta+1}}{\beta(\beta+1)} R_1$$

$$= \int_0^\pi \psi_1(t) S_{\beta-1}(\omega t)\, dt$$

$$= [-t\psi_2(t) S_{\beta-1}(\omega t)]_0^\pi + (\beta-1) \int_0^\pi \psi_2(t) S_{\beta-2}(\omega t)\, dt$$

[since $t\psi_2(t) \to 0$ as $t \to 0$*]

$$= (\beta-1) \int_0^\pi \psi_2(t) S_{\beta-2}(\omega t)\, dt = \ldots$$

$$= (\beta-1)(\beta-2)\ldots(\beta-h) \int_0^\pi \psi_{h+1}(t) S_{\beta-h-1}(\omega t)\, dt$$

$$= \frac{(\beta-1)(\beta-2)\ldots(\beta-h)}{\Gamma(h+1-\beta)} \int_0^\pi S_{\beta-h-1}(\omega t)\, dt \int_t^\pi \left(\log\frac{u}{t}\right)^{h-\beta} \psi_\beta(u) \frac{du}{u}$$

$$= \frac{(\beta-1)(\beta-2)\ldots(\beta-h)}{\Gamma(h+1-\beta)} \int_0^\pi \psi_\beta(u) \frac{du}{u} \int_0^u \left(\log\frac{u}{t}\right)^{h-\beta} S_{\beta-h-1}(\omega t)\, dt$$

$$= \Gamma(\beta) \int_0^\pi \psi_\beta(u) S_0(\omega u)\, du = \frac{\Gamma(\beta)}{\omega} \int_0^\pi \psi_\beta(u) \frac{1-\cos\omega u}{u}\, du,$$

by (1.3) and (2.2). If β is a positive integer, we obtain the same formula after integrating by parts $\beta-1$ times. Hence, if (1.5) is satisfied, we have, as in the proof of Theorem A,

$$R_1 = o\left(\frac{1}{\omega}\right) \quad (\omega \to \infty).$$

Similarly, by (1.5),

$$\frac{\pi(\log\omega)^{\beta+2}}{(\beta+1)^2} R_2 = \int_0^\pi \psi_1(t) S_\beta(\omega t)\, dt$$

$$= O\left(\int_0^\pi |\psi_{\beta+1}(t)| \frac{1-\cos\omega t}{\omega t}\, dt\right) = o\left(\frac{1}{\omega}\right).$$

* See Hardy (2).

Therefore
$$\bar{R}'_{\beta+1}(\omega) = \frac{D}{(\beta+2)\pi\omega} + o\left(\frac{1}{\omega}\right);$$

and
$$\bar{R}_{\beta+1}(\lambda\omega) - \bar{R}_{\beta+1}(\omega) = \int_{\omega}^{\lambda\omega} \bar{R}'_{\beta+1}(t)\,dt$$
$$= \frac{D}{(\beta+2)\pi} \log\lambda + o(1),$$

which proves the Theorem B.

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WELL-POISED BASIC HYPERGEOMETRIC SERIES

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1. Introduction

DOUGALL's theorem for the sum of a well-poised hypergeometric series ${}_7F_6$ of the ordinary type was given in 1907. In 1932, I gave in this *Journal** the formula

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix}a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & f; \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f\end{matrix}\right] \\
&= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(b-a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)}\times \\
&\quad\times\frac{\Gamma(b+c-a)\Gamma(b+d-a)\Gamma(b+e-a)\Gamma(b+f-a)}{\Gamma(1+a-c-f)\Gamma(1+a-d-f)\Gamma(1+a-e-f)}- \\
&-\frac{\Gamma(1+2b-a)\Gamma(b+c-a)\Gamma(b+d-a)\Gamma(b+e-a)\Gamma(b+f-a)}{\Gamma(1+b-c)\Gamma(1+b-d)\Gamma(1+b-e)\Gamma(1+b-f)}\times \\
&\quad\times\frac{\Gamma(a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(b-a)\Gamma(1+a)\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(f)}\times \\
&\quad\times{}_7F_6\left[\begin{matrix}2b-a, & 1+b-\frac{1}{2}a, & b, & b+c-a, & b+d-a, \\ & b-\frac{1}{2}a, & 1+b-a, & 1+b-c, & 1+b-d,\end{matrix}\right. \\
&\qquad\qquad\qquad\qquad\left.\begin{matrix}b+e-a, & b+f-a; \\ 1+b-e, & 1+b-f\end{matrix}\right], \qquad (1.1)
\end{aligned}
$$

where $1+2a = b+c+d+e+f$.

When one of the parameters c, d, e, f is a negative integer, this reduces to Dougall's theorem.

In the same paper I obtained a formula† connecting four well-poised series of the type ${}_9F_8$. If we write

$$
\begin{aligned}
V(a;b,c,d,e,f,g,h) &= \frac{\Gamma(1+a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(f)\Gamma(g)\Gamma(h)}{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\times \\ \times\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-h)}\times \\
&\times{}_9F_8\left[\begin{matrix}a, & 1+\frac{1}{2}a, & b, & c, & d, & e, \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e,\end{matrix}\right. \\
&\qquad\qquad\left.\begin{matrix}f, & g, & h; \\ 1+a-f, & 1+a-g, & 1+a-h\end{matrix}\right],
\end{aligned}
$$

* Bailey, **2** (7.4). See also Bailey, **1**, § 6.6 (3).

† Bailey, **2**, § 9. For the form given here see Bailey, **1**, § 6.8 (3).

the formula is

$$\operatorname{cosec}(b-a)\pi\{V(a;b,c,d,e,f,g,h)-$$
$$-V(2b-a;b,b+c-a,b+d-a,b+e-a,b+f-a,b+g-a,b+h-a)\}$$
$$=\frac{\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(f+b-a)\Gamma(g+b-a)}{\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)}\times$$
$$\times\frac{\Gamma(h+b-a)\operatorname{cosec}(1+a-f-g-h)\pi}{\Gamma(1+a-g-h)\Gamma(1+a-f-h)\Gamma(1+a-f-g)}\times$$
$$\times\{V(1+2a-c-d-e;b,1+a-d-e,1+a-c-e,1+a-c-d,f,g,h)-$$
$$-V(2b-2a-1+c+d+e;b,b-a+c,b-a+d,b-a+e,$$
$$1+a-g-h,1+a-f-h,1+a-f-g)\},\quad(1.2)$$

provided that $$2+3a=b+c+d+e+f+g+h.\quad(1.3)$$

By iterating this formula, I showed* that

$$\operatorname{cosec}(b-a)\pi\{V(a;b,c,d,e,f,g,h)-$$
$$-V(2b-a;b,b+c-a,b+d-a,b+e-a,b+f-a,b+g-a,b+h-a)\}$$
$$=\frac{\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(f)\Gamma(c+b-a)\Gamma(d+b-a)}{\Gamma(1+a-c-g)\Gamma(1+a-d-g)\Gamma(1+a-e-g)\Gamma(1+a-f-g)}\times$$
$$\times\frac{\Gamma(e+b-a)\Gamma(f+b-a)\operatorname{cosec}(g-h)\pi}{\Gamma(1+a-c-h)\Gamma(1+a-d-h)\Gamma(1+a-e-h)\Gamma(1+a-f-h)}\times$$
$$\times\{V(b-g+h;b,1+a-c-g,1+a-d-g,1+a-e-g,$$
$$1+a-f-g,h,h+b-a)-$$
$$-V(b+g-h;b,1+a-c-h,1+a-d-h,1+a-e-h,$$
$$1+a-f-h,g,g+b-a)\},\quad(1.4)$$

provided that the condition (1.3) is satisfied.

In (1.2) and (1.4) any one of the series ${}_9F_8$ is of general well-poised type, except for the second parameter and the restriction that the sum of the denominator parameters exceeds that of the numerator parameters by two. When h is a negative integer, these formulae reduce to formulae connecting two terminating well-poised series ${}_9F_8$.†

It is now twenty-five years since Jackson‡ gave the basic analogue

* Bailey, **1,** § 7.6 (2). † Bailey, **1,** § 4.3 (7) and § 7.6 (1).

‡ Jackson, **4.** See also Bailey, **1,** § 8.3 (1).

of Dougall's theorem. In 1929 Watson gave a transformation of a terminating well-poised ${}_8\Phi_7$ into a Saalschützian ${}_4\Phi_3$, corresponding to Whipple's transformation of a well-poised ${}_7F_6$, and in the same year I gave a transformation* of a terminating well-poised ${}_{10}\Phi_9$ which is exactly analogous to the particular case of (1.2) when h is a negative integer, and gives a generalization† of Watson's transformation of an ${}_8\Phi_7$ when the series does not terminate.

It has thus been obvious since 1932 that formulae corresponding to (1.1), (1.2), and (1.4) are probably true for basic series. All these formulae, however, were discovered by means of contour integrals of Barnes's type, and it did not appear obvious how to attack the problem for basic series. I now find that the basic analogue of (1.1) is an immediate result of a formula which I published in this *Journal* ten years ago.‡

In the present paper I give the basic analogues of (1.1), (1.2), and (1.4). The proof of the analogue of (1.2) was suggested by the method of obtaining transformations of integrals of Barnes's type§ by which (1.2) was discovered. This proof, when written out for (1.2) itself, consists in using the series equivalent to the integral of Barnes's type, but the details both for (1.2) itself and its analogue were found to be surprisingly troublesome.

2. Notation

In this paper I use the notation

$$(a)_n = (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), \quad (a)_0 = 1,$$

where $|q| < 1$.

I also use the function $\Pi(a)$ defined by

$$1/\Pi(a) = \prod_{n=0}^{\infty} (1-aq^n).$$

3. The basic analogue of (1.1)

If

$$\chi[a;b,c,d,e,f] = \frac{\Pi(aq)}{\Pi(aq/b)\Pi(aq/c)\Pi(aq/d)\Pi(aq/e)\Pi(aq/f)\Pi(a^2q^2/bcdef)} \times$$

$$\times {}_8\Phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f; \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix} \; a^2q^2/bcdef\right],$$

* See Bailey, **1**, § 8.5 (1).

† Bailey, **1**, § 8.5 (3).

‡ Bailey, **3**.

§ See Bailey, **1**, § 6.7, or **2**, § 8.

then it is known that*

$$\frac{\chi[a;b,c,d,e,f]}{\Pi(aq/def)\Pi(def/a)\Pi(bd/a)\Pi(be/a)\Pi(q/c)\Pi(bf/a)}$$

$$=\frac{\chi[ef/c;e,f,aq/bc,aq/cd,ef/a]}{\Pi(aq/b)\Pi(b/a)\Pi(aq/ef)\Pi(aq/df)\Pi(aq/de)\Pi(a^2q^2/bcdef)}+$$

$$+\frac{b\chi[b^2/a;b,bc/a,bd/a,be/a,bf/a]}{a\Pi(aq/bc)\Pi(d)\Pi(e)\Pi(f)\Pi(a^2q/bdef)\Pi(bdef/a^2)}. \tag{3.1}$$

If we take $a^2q = bcdef$, the first series ${}_8\Phi_7$ on the right-hand side of (3.1) reduces to a ${}_6\Phi_5$ which can be summed by the formula†

$${}_6\Phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d; \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, \end{matrix}\ aq/bcd\right]$$

$$=\frac{\Pi(aq/b)\Pi(aq/c)\Pi(aq/d)\Pi(aq/bcd)}{\Pi(aq)\Pi(aq/bc)\Pi(aq/bd)\Pi(aq/cd)}. \tag{3.2}$$

We thus find that

$${}_8\Phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f; \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}\ q\right]$$

$$=\frac{\Pi(aq/c)\Pi(aq/d)\Pi(aq/e)\Pi(aq/f)\Pi(bc/a)\Pi(bd/a)\Pi(be/a)\Pi(bf/a)}{\Pi(aq)\Pi(b/a)\Pi(aq/de)\Pi(aq/ce)\Pi(aq/cd)\Pi(aq/cf)\Pi(aq/df)\Pi(aq/ef)}-$$

$$-\frac{\Pi(b^2q/a)\Pi(bc/a)\Pi(bd/a)\Pi(be/a)\Pi(bf/a)\times \\ \times\Pi(aq/c)\Pi(aq/d)\Pi(aq/e)\Pi(aq/f)\Pi(a/b)}{\Pi(bq/c)\Pi(bq/d)\Pi(bq/e)\Pi(bq/f)\Pi(aq)\Pi(c)\Pi(d)\Pi(e)\Pi(f)\Pi(b/a)}\times$$

$$\times\,{}_8\Phi_7\left[\begin{matrix} b^2/a, & qb/\sqrt{a}, & -qb/\sqrt{a}, & b, & bc/a, & bd/a, & be/a, & bf/a; \\ & b/\sqrt{a}, & -b/\sqrt{a}, & bq/a, & bq/c, & bq/d, & bq/e, & bq/f \end{matrix}\ q\right],$$

where $a^2q = bcdef$. (3.3)

This is the analogue of (1.1).

4. An alternative form of (3.3)

Since $\Pi(aq^m) = (a)_m\Pi(a)$, we can write (3.3) in the form‡

$$\sum_{m=0}^{\infty}\frac{\Pi(aq^m)\Pi(q^{m+1}\sqrt{a})\Pi(-q^{m+1}\sqrt{a})}{(q)_m\Pi(q^m\sqrt{a})\Pi(-q^m\sqrt{a})\Pi(aq^{m+1}/b)}\times$$

$$\times\frac{\Pi(bq^m)\Pi(cq^m)\Pi(dq^m)\Pi(eq^m)\Pi(fq^m)q^m}{\Pi(aq^{m+1}/c)\Pi(aq^{m+1}/d)\Pi(aq^{m+1}/e)\Pi(aq^{m+1}/f)}$$

* Bailey, **3** (5.1).

† This is the particular case of Bailey, **1**, § 8.5 (3), when $aq = de$.

‡ Corresponding to Bailey, **1**, § 6.6 (2).

$$=\frac{\Pi(b)\Pi(c)\Pi(d)\Pi(e)\Pi(f)\Pi(bc/a)\Pi(bd/a)\Pi(be/a)\Pi(bf/a)}{\Pi(aq/de)\Pi(aq/ce)\Pi(aq/cd)\Pi(aq/cf)\times \times\Pi(aq/df)\Pi(aq/ef)\Pi(b/a)\Pi(aq/b)}+$$

$$+\frac{b}{a}\sum_{m=0}^{\infty}\frac{\Pi(b^2q^m/a)\Pi(bq^{m+1}/\sqrt{a})\Pi(-bq^{m+1}/\sqrt{a})\Pi(bq^m)}{(q)_m\Pi(bq^m/\sqrt{a})\Pi(-bq^m/\sqrt{a})\Pi(bq^{m+1}/a)}\times$$

$$\times\frac{\Pi(bcq^m/a)\Pi(bdq^m/a)\Pi(beq^m/a)\Pi(bfq^m/a)q^m}{\Pi(bq^{m+1}/c)\Pi(bq^{m+1}/d)\Pi(bq^{m+1}/e)\Pi(bq^{m+1}/f)}. \quad (4.1)$$

Now replace a, c, d, e, k, b by k, kc/a, kd/a, ke/a, a, at, where, as usual, $k=a^2q/cde$, and we get*

$$\frac{\Pi(at)\Pi(ct)\Pi(dt)\Pi(et)\Pi(1/t)}{\Pi(atq/c)\Pi(atq/d)\Pi(atq/e)\Pi(at/k)\Pi(kq/at)}$$

$$=\frac{\Pi(c)\Pi(d)\Pi(e)}{\Pi(kc/a)\Pi(kd/a)\Pi(ke/a)\Pi(a/k)}\times$$

$$\times\left[\sum_{m=0}^{\infty}\frac{q^m\Pi(kq^m)\Pi(q^{m+1}\sqrt{k})\Pi(-q^{m+1}\sqrt{k})\Pi(atq^m)}{(q)_m\Pi(q^m\sqrt{k})\Pi(-q^m\sqrt{k})\Pi(kq^{m+1}/at)\Pi(aq^{m+1}/c)}\times\right.$$

$$\times\frac{\Pi(kcq^m/a)\Pi(kdq^m/a)\Pi(keq^m/a)\Pi(q^m/t)}{\Pi(aq^{m+1}/d)\Pi(aq^{m+1}/e)\Pi(ktq^{m+1})}-$$

$$-\frac{at}{k}\sum_{m=0}^{\infty}\frac{q^m\Pi(a^2t^2q^m/k)\Pi(atq^{m+1}/\sqrt{k})\Pi(-atq^{m+1}/\sqrt{k})}{\Pi(atq^m/\sqrt{k})\Pi(-atq^m/\sqrt{k})\Pi(at^2q^{m+1})}\times$$

$$\left.\times\frac{\Pi(aq^m/k)\Pi(ctq^m)\Pi(dtq^m)\Pi(etq^m)\Pi(atq^m)}{\Pi(detq^m)\Pi(cetq^m)\Pi(cdtq^m)\Pi(atq^{m+1}/k)}\right]. \quad (4.2)$$

If $t=q^n$, where n is a positive integer, this reduces to

$$\frac{\Pi(aq^n)\Pi(cq^n)\Pi(dq^n)\Pi(eq^n)}{\Pi(aq^{n+1}/c)\Pi(aq^{n+1}/d)\Pi(aq^{n+1}/e)\Pi(aq^n/k)\Pi(kq^{1-n}/a)}$$

$$=\frac{\Pi(c)\Pi(d)\Pi(e)}{\Pi(kc/a)\Pi(kd/a)\Pi(ke/a)\Pi(a/k)}\times$$

$$\times\sum_{m=0}^{n}\frac{q^m\Pi(kq^m)\Pi(q^{m+1}\sqrt{k})\Pi(-q^{m+1}\sqrt{k})\Pi(aq^{n+m})}{(q)_m\Pi(q^m\sqrt{k})\Pi(-q^m\sqrt{k})\Pi(kq^{m-n+1}/a)\Pi(aq^{m+1}/c)}\times$$

$$\times\frac{\Pi(kcq^m/a)\Pi(kdq^m/a)\Pi(keq^m/a)(q^{-n})_m}{\Pi(aq^{m+1}/d)\Pi(aq^{m+1}/e)\Pi(kq^{m+n+1})}. \quad (4.3)$$

This formula is, of course, equivalent to Jackson's analogue of Dougall's theorem.

* Corresponding to Bailey, **1**, § 6.7.

5. A transformation from (4.3)

Using the identities

$$\Pi(Aq^{-p}) = (-1)^p q^{\frac{1}{2}p(p+1)}\Pi(A)/A^p(q/A)_p,$$

$$\Pi(Aq^n)\Pi(q^{1-n}/A) = \Pi(A)\Pi(q/A)(-1)^n A^n q^{\frac{1}{2}n(n-1)},$$

we get from (4.3)

$$\sum_{n=0}^{\infty} \frac{\Pi(aq^n)\Pi(cq^n)\Pi(dq^n)\Pi(eq^n)\Pi(\rho_1 q^n)\Pi(\rho_2 q^n)x^n}{(q)_n\Pi(aq^{n+1}/c)\Pi(aq^{n+1}/d)\Pi(aq^{n+1}/e)\Pi(\sigma_1 q^n)\Pi(aq^{n+1}/b)}$$

$$= \frac{\Pi(c)\Pi(d)\Pi(e)\Pi(kq/a)}{\Pi(kc/a)\Pi(kd/a)\Pi(ke/a)} \times$$

$$\times \sum_{n=0}^{\infty}\sum_{n=0}^{n} \frac{q^m\Pi(kq^m)\Pi(q^{m+1}\sqrt{k})\Pi(-q^{m+1}\sqrt{k})\Pi(aq^{n+m})\Pi(kcq^m/a)}{(q)_m\Pi(q^m\sqrt{k})\Pi(-q^m\sqrt{k})\Pi(kq^{m-n+1}/a)\Pi(aq^{m+1}/c)} \times$$

$$\times \frac{\Pi(kdq^m/a)\Pi(keq^m/a)(q^{-n})_m}{\Pi(aq^{m+1}/d)\Pi(aq^{m+1}/e)\Pi(kq^{m+n+1})} \times$$

$$\times \frac{\Pi(\rho_1 q^n)\Pi(\rho_2 q^n)x^n}{(q)_n\Pi(\sigma_1 q^n)\Pi(aq^{n+1}/b)}(-1)^n\left(\frac{a}{k}\right)^n q^{\frac{1}{2}n(n-1)}.$$

Writing $n = m+p$, and using the identity

$$(q^{-m-p})_m = \frac{(-1)^m(q)_{m+p}}{(q)_p\, q^{\frac{1}{2}m(2p+m+1)}},$$

the right-hand side becomes

$$\frac{\Pi(c)\Pi(d)\Pi(e)}{\Pi(kc/a)\Pi(kd/a)\Pi(ke/a)}$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi(kq^m)\Pi(q^{m+1}\sqrt{k})\Pi(-q^{m+1}\sqrt{k})\Pi(aq^{2m})\Pi(kcq^m/a)\Pi(kdq^m/a)}{(q)_m\Pi(q^m\sqrt{k})\Pi(-q^m\sqrt{k})\Pi(aq^{m+1}/c)\Pi(aq^{m+1}/d)} \times$$

$$\times \frac{\Pi(keq^m/a)}{\Pi(aq^{m+1}/e)\Pi(kq^{2m+1})}\,\frac{\Pi(\rho_1 q^m)\Pi(\rho_2 q^m)}{\Pi(\sigma_1 q^m)\Pi(aq^{m+1}/b)}\left(-\frac{ax}{k}\right)^m \times$$

$$\times \Phi\begin{bmatrix} aq^{2m}, & a/k, & \rho_1 q^m, & \rho_2 q^m; & \\ & kq^{2m+1}, & \sigma_1 q^m, & aq^{m+1}/b & x \end{bmatrix}.$$

In this result there may be any number of the symbols ρ, σ. In particular we find that

$$\sum_{n=0}^{\infty} \frac{\Pi(aq^n)\Pi(q^{n+1}\sqrt{a})\Pi(-q^{n+1}\sqrt{a})\Pi(bq^n)\Pi(cq^n)\Pi(dq^n)}{(q)_n\Pi(q^n\sqrt{a})\Pi(-q^n\sqrt{a})\Pi(aq^{n+1}/b)\Pi(aq^{n+1}/c)\Pi(aq^{n+1}/d)} \times$$

$$\times \frac{\Pi(eq^n)\Pi(fq^n)\Pi(gq^n)\Pi(hq^n)}{\Pi(aq^{n+1}/e)\Pi(aq^{n+1}/f)\Pi(aq^{n+1}/g)\Pi(aq^{n+1}/h)} q^n$$

$$= \frac{\Pi(c)\Pi(d)\Pi(e)}{\Pi(kc/a)\Pi(kd/a)\Pi(ke/a)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Pi(kq^m)\Pi(q^{m+1}\sqrt{k})\Pi(-q^{m+1}\sqrt{k}) \times \times \Pi(aq^{2m+1})\Pi(kcq^m/a)\Pi(kdq^m/a)}{(q)_m \Pi(q^m\sqrt{k})\Pi(-q^m\sqrt{k})\Pi(aq^{m+1}/c)\Pi(aq^{m+1}/d)} \times$$

$$\times \frac{\Pi(keq^m/a)\Pi(bq^m)\Pi(fq^m)\Pi(gq^m)\Pi(hq^m)(aq/k)^m}{\Pi(aq^{m+1}/e)\Pi(aq^{m+1}/b)\Pi(aq^{m+1}/f)\Pi(aq^{m+1}/g)\Pi(aq^{m+1}/h)} \times$$

$$\times {}_8\Phi_7\left[\begin{matrix} aq^{2m}, & q^{m+1}\sqrt{a}, & -q^{m+1}\sqrt{a}, & bq^m, & fq^m, & gq^m, & hq^m, & a/k; \\ & q^m\sqrt{a}, & -q^m\sqrt{a}, & aq^{m+1}/b, & aq^{m+1}/f, & aq^{m+1}/g, & aq^{m+1}/h, & kq^{2m+1} \end{matrix}\; q\right]. \tag{5.1}$$

The series on the left of (5.1) is, of course, a multiple of a well-poised series ${}_{10}\Phi_9$, with numerator parameters a, $q\sqrt{a}$, $-q\sqrt{a}$, b, c, d, e, f, g, h. The change in the order of summation is easily justified because of the powers of q that occur.

6. A transformation from (4.2)

Now in (4.2) take $t = bq^n/a$, and we obtain the sum

$$\sum_{n=0}^{\infty} \frac{\Pi(bq^n)\Pi(bcq^n/a)\Pi(bdq^n/a)\Pi(beq^n/a)\Pi(\rho_1 bq^n/a)\Pi(\rho_2 bq^n/a)}{(q)_n \Pi(bq^{n+1}/c)\Pi(bq^{n+1}/d)\Pi(bq^{n+1}/e)\Pi(bq^{n+1}/a)\Pi(\sigma_1 bq^n/a)} y^n$$

as the sum of two double series. In the first we simply change the order of summation and obtain

$$\frac{\Pi(c)\Pi(d)\Pi(e)\Pi(b/k)\Pi(qk/b)}{\Pi(kc/a)\Pi(kd/a)\Pi(ke/a)\Pi(a/k)\Pi(bq/a)\Pi(a/b)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{q^m \Pi(kq^m)\Pi(q^{m+1}\sqrt{k})\Pi(-q^{m+1}\sqrt{k})\Pi(bq^m)\Pi(kcq^m/a)}{(q)_m \Pi(q^m\sqrt{k})\Pi(-q^m\sqrt{k})\Pi(kq^{m+1}/b)\Pi(aq^{m+1}/c)} \times$$

$$\times \frac{\Pi(kdq^m/a)\Pi(keq^m/a)\Pi(aq^m/b)\Pi(\rho_1 b/a)\Pi(\rho_2 b/a)}{\Pi(aq^{m+1}/d)\Pi(aq^{m+1}/e)\Pi(kbq^{m+1}/a)\Pi(\sigma_1 b/a)} \times$$

$$\times \Phi\left[\begin{matrix} bq^m, & bq^{-m}/k, & \rho_1 b/a, & \rho_2 b/a; \\ & bq^{1-m}/a, & kbq^{m+1}/a, & \sigma_1 b/a \end{matrix}\; y\right].$$

In the second double series on the right we put $m = p-n$ and then change the order of summation, and we obtain

$$-\frac{b}{k}\frac{\Pi(c)\Pi(d)\Pi(e)\Pi(b/k)\Pi(qk/b)}{\Pi(kc/a)\Pi(kd/a)\Pi(ke/a)\Pi(a/k)\Pi(bq/a)\Pi(a/b)}\times$$
$$\times\sum_{p=0}^{\infty}\frac{q^p\Pi(b^2q^p/k)\Pi(bq^{p+1}/\sqrt{k})\Pi(-bq^{p+1}/\sqrt{k})\Pi(aq^p/k)}{(q)_p\Pi(bq^p/\sqrt{k})\Pi(-bq^p/\sqrt{k})\Pi(b^2q^{p+1}/a)}\times$$
$$\times\frac{\Pi(bcq^p/a)\Pi(bdq^p/a)\Pi(beq^p/a)\Pi(bq^p)\Pi(\rho_1 b/a)\Pi(\rho_2 b/a)}{\Pi(bdeq^p/a)\Pi(bceq^p/a)\Pi(bcdq^p/a)\Pi(bq^{p+1}/a)\Pi(\sigma_1 b/a)}\times$$
$$\times\Phi\begin{bmatrix}\rho_1 b/a, & \rho_2 b/a, & b^2q^p/k, & q^{-p}; & \\ & \sigma_1 b/a, & kq^{1-p}/a, & b^2q^{p+1}/a & y\end{bmatrix}.$$

Again there may be any number of the symbols ρ, σ. In particular we have

$$\sum_{n=0}^{\infty}\frac{\Pi(b^2q^n/a)\Pi(q^{n+1}b/\sqrt{a})\Pi(-q^{n+1}b/\sqrt{a})\Pi(bq^n)\Pi(bcq^n/a)}{(q)_n\Pi(q^nb/\sqrt{a})\Pi(-q^nb/\sqrt{a})\Pi(bq^{n+1}/a)\Pi(bq^{n+1}/c)}\times$$
$$\times\frac{\Pi(bdq^n/a)\Pi(beq^n/a)\Pi(bfq^n/a)\Pi(bgq^n/a)\Pi(bhq^n/a)q^n}{\Pi(bq^{n+1}/d)\Pi(bq^{n+1}/e)\Pi(bq^{n+1}/f)\Pi(bq^{n+1}/g)\Pi(bq^{n+1}/h)}$$
$$=\frac{\Pi(c)\Pi(d)\Pi(e)\Pi(b/k)\Pi(qk/b)\Pi(b^2q/a)\Pi(bf/a)\Pi(bg/a)\Pi(bh/a)}{\Pi(kc/a)\Pi(kd/a)\Pi(ke/a)\Pi(a/k)\Pi(a/b)\Pi(bq/f)\Pi(bq/g)\Pi(bq/h)}\times$$
$$\times\Bigg\{\sum_{m=0}^{\infty}\frac{q^m\Pi(kq^m)\Pi(q^{m+1}\sqrt{k})\Pi(-q^{m+1}\sqrt{k})\Pi(bq^m)}{(q)_m\Pi(q^m\sqrt{k})\Pi(-q^m\sqrt{k})\Pi(kq^{m+1}/b)\Pi(aq^{m+1}/c)}\times$$
$$\times\frac{\Pi(kcq^m/a)\Pi(kdq^m/a)\Pi(keq^m/a)\Pi(aq^m/b)}{\Pi(aq^{m+1}/d)\Pi(aq^{m+1}/e)\Pi(kbq^{m+1}/a)}\times$$
$$\times{}_8\Phi_7\begin{bmatrix}b^2/a, & qb/\sqrt{a}, & -qb/\sqrt{a}, & bq^m, & bq^{-m}/k, & bf/a, & bg/a, & bh/a; & \\ & b\sqrt{a}, & -b\sqrt{a}, & bq^{1-m}/a, & kbq^{m+1}/a, & bq/f, & bq/g, & bq/h & q\end{bmatrix}-$$
$$-\frac{b}{k}\sum_{p=0}^{\infty}\frac{q^p\Pi(b^2q^p/k)\Pi(bq^{p+1}/\sqrt{k})\Pi(-bq^{p+1}/\sqrt{k})\Pi(aq^p/k)}{(q)_p\Pi(bq^p/\sqrt{k})\Pi(-bq^p/\sqrt{k})\Pi(b^2q^{p+1}/a)}\times$$
$$\times\frac{\Pi(bcq^p/a)\Pi(bdq^p/a)\Pi(beq^p/a)\Pi(bq^p)}{\Pi(bdeq^p/a)\Pi(bceq^p/a)\Pi(bcdq^p/a)\Pi(bq^{p+1}/a)}\times$$
$$\times{}_8\Phi_7\begin{bmatrix}b^2/a, & qb/\sqrt{a}, & -qb/\sqrt{a}, & bf/a, & bg/a, & bh/a, & b^2q^p/k, & q^{-p}; & \\ & b/\sqrt{a}, & -b/\sqrt{a}, & bq/f, & bq/g, & bq/h, & kq^{1-p}/a, & b^2q^{p+1}/a & q\end{bmatrix}\Bigg\}. \tag{6.1}$$

7. The basic analogue of (1.2)

If
$$a^3q^2 = bcdefgh, \tag{7.1}$$
the second series ${}_8\Phi_7$ on the right of (6.1) can be summed by Jackson's analogue of Dougall's theorem. Also, under this condition, the first series ${}_8\Phi_7$ on the right of (6.1) can be combined with the series ${}_8\Phi_7$ on the right of (5.1) so that we can use (3.3) with a, b, c, d, e, f replaced by $aq^{2m}, bq^m, fq^m, gq^m, hq^m, a/k$.

Writing
$$W(a;b,c,d,e,f,g,h)$$
$$= \frac{\Pi(aq)\Pi(b)\Pi(c)\Pi(d)\Pi(e)\Pi(f)\Pi(g)\Pi(h)}{\Pi(aq/b)\Pi(aq/c)\Pi(aq/d)\Pi(aq/e)\Pi(aq/f)\Pi(aq/g)\Pi(aq/h)}\times$$
$$\times {}_{10}\Phi_9\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f, & g, & h; \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f, & aq/g, & aq/h, \end{matrix} q\right],$$
we find, after considerable reduction, that
$$\Pi(b/a)\Pi(qa/b)[W(a;b,c,d,e,f,g,h)-$$
$$-(b/a)W(b^2/a;b,bc/a,bd/a,be/a,bf/a,bg/a,bh/a)]$$
$$= \frac{\Pi(c)\Pi(d)\Pi(e)\Pi(bf/a)\Pi(bg/a)\Pi(bh/a)\Pi(b/k)\Pi(qk/b)}{\Pi(kc/a)\Pi(kd/a)\Pi(ke/a)\Pi(aq/gh)\Pi(aq/fh)\Pi(aq/fg)}\times$$
$$\times[W(k;b,kc/a,kd/a,ke/a,f,g,h)-$$
$$-(b/k)W(b^2/k;b,bc/a,bd/a,be/a,bf/k,bg/k,bh/k)], \tag{7.2}$$
where $k = a^2q/cde$, and $a^3q^2 = bcdefgh$.

This is the analogue of (1.2).

8. The basic analogue of (1.4)

By iterating (7.2) we obtain a formula equivalent to
$$\Pi(b/a)\Pi(qa/b)[W(a;b,c,d,e,f,g,h)-$$
$$-(b/a)W(b^2/a;b,bc/a,bd/a,be/a,bf/a,bg/a,bh/a)]$$
$$= \frac{\Pi(c)\Pi(d)\Pi(e)\Pi(f)\Pi(bc/a)\Pi(bd/a)\Pi(be/a)\Pi(bf/a)}{\Pi(aq/cg)\Pi(aq/dg)\Pi(aq/eg)\Pi(aq/fg)\Pi(aq/ch)\times \\ \times\Pi(aq/dh)\Pi(aq/eh)\Pi(aq/fh)}\times$$
$$\times\Pi(g/h)\Pi(qh/g)[W(bh/g;b,aq/cg,aq/dg,aq/eg,aq/fg,bh/a,h)-$$
$$-(g/h)W(bg/h;b,aq/ch,aq/dh,aq/eh,aq/fh,bg/a,g)], \tag{8.1}$$
where again $a^3q^2 = bcdefgh$. This is the analogue of (1.4).

9. Conclusion

From (7.2) and (8.1), when one of the parameters is of the form q^{-n}, where n is a positive integer, we can deduce relations connecting two or three well-poised series of the type ${}_{10}\Phi_9$. By letting $n \to \infty$ we obtain relations connecting two or three series of the type ${}_8\Phi_7$ in which the parameters are unrestricted, apart from convergence conditions and the special forms of the second and third numerator parameters. My transformations of terminating ${}_{10}\Phi_9$ mentioned in § 1 are particular cases of (7.2) and (8.1). One of these contains Jackson's analogue of Dougall's theorem as an obvious particular case, just as Dougall's theorem itself can be deduced from (1.2). Thus (7.2) and (8.1) include as particular or limiting cases all the known results for well-poised basic series.

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THE TWO-DIMENSIONAL AEROFOIL IN A BOUNDED STREAM

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1. THE two-dimensional perfect-fluid theory for finding the lift and moment forces acting on an aerofoil in an unlimited fluid is well known for a wide variety of aerofoil shapes. Very few exact solutions exist when the fluid is limited by one or more boundaries or when the aerofoil is near other bodies. The forces acting on a flat-plate aerofoil which is in a stream bounded by one or two straight parallel walls have been found by Tomotika and others (**5–12**) using conformal transformations. The forces on a circular-arc aerofoil in the presence of a plane wall were found by conformal transformations by the present writer (**1**). Tomotika has shown that a series form of solution for the flat-plate problems may be deduced from his exact analysis, the series being very suitable for numerical computation provided that the plate is not too near the wall. The flat-plate problems have been re-examined by Havelock (**2**) by quite a different method which leads directly to expansions for the lift and moment forces acting on the plate. His analysis actually applies to an elliptical aerofoil of which the flat plate is a limiting case. The method used, however, appears to be somewhat complicated, and it is of interest to re-examine the problem once more since the series solution can be obtained fairly easily by using quite elementary ideas. Moreover, the present method applies not only to the elliptical and flat-plate aerofoils but also to the usual more general types of aerofoils. In § 2, therefore, the problem of the fairly general aerofoil in the presence of a plane wall is discussed, and the extension to an aerofoil in a tunnel bounded by two parallel walls is indicated in § 7.

In discussing the effect of the ground on the lift of a monoplane aerofoil Tomotika (**10**) found that his results for a flat plate were in qualitative agreement with experiment. The results for a circular-arc aerofoil differed considerably in some cases from those for a flat-plate and from experiment (**1**). The work of § 2 when applied to the circular-arc aerofoil shows more clearly why the flat-plate and circular-arc aerofoils are different, and it also enables us to see how the thickness of the aerofoil affects the results. Only three terms of the series

expansion for the lift acting on the general aerofoil in the presence of a plane wall are given in the present paper so that results obtained from these terms must be accepted with caution. It appears, however, that, for large distances of the aerofoil from the ground, camber and thickness both add to the decrease in the lift which is found on a flat plate compared with the lift in an unbounded stream. As the flat plate gets much nearer the ground it is known that, for small angles of attack, the lift eventually increases above the value found in an unbounded stream. In this case the present work indicates that the effects of camber and thickness on the flat-plate results appear to act mainly in opposite directions so that we might expect that the flat plate would give a reasonable approximation to actual aerofoil shapes.

Havelock (**2**) also obtained expansions for the forces on a flat plate when the stream is bounded by parallel free surfaces, or by one rigid and one free surface, by taking the boundary condition in an approximate form. The methods of the present paper could be used for similar problems when the aerofoil has a more general shape.

In § 7 the method of solution is indicated for the problem of an infinite cascade of similar aerofoils in a uniform stream, a problem of some interest in airscrew theory. The exact solution is known for a cascade of flat-plate aerofoils but not for other shapes (**13**).

2. Consider the two-dimensional motion due to a cylinder whose cross-section is the curve $\eta = 0$, $\xi = 0 \to -2\pi$ in the transformation

$$z = e^{-i\zeta} \sum_{n=0}^{\infty} a_n e^{in\zeta} \quad (\zeta = \xi + i\eta) \tag{2.1}$$

placed in a uniform stream bounded by one plane wall $y = -b$. The origin of coordinates is inside the cross-section C of the cylinder. If c is the stream velocity parallel to the negative direction of the x-axis and if a circulation $K = 2\pi A_0$ also exists round the cylinder, a suitable complex potential function w which makes the boundary wall a stream-line is

$$w = cz + iA_0\{\log z - \log(z+2ib)\} + \sum_{n=1}^{\infty} \left\{\frac{A_n}{z^n} + \frac{\bar{A}_n}{(z+2ib)^n}\right\}, \tag{2.2}$$

where a bar placed over a quantity denotes the complex conjugate

of that quantity. Apart from a constant, equation (2.2) may be written in the form

$$w = cz+iA_0\log z+\sum_{n=1}^{\infty}\frac{A_n}{z^n}+\sum_{n=1}^{\infty}B_n z^n, \tag{2.3}$$

where

$$B_n = \frac{{}^n\alpha_0 A_0}{b^n}+\sum_{r=1}^{\infty}\frac{{}^n\alpha_r \bar{A}_r}{b^{n+r}}, \tag{2.4}$$

and

$${}^n\alpha_0 = \frac{i^{n+1}}{n2^n}, \qquad {}^n\alpha_r = \binom{n+r-1}{n}\frac{i^{n-r}}{2^{n+r}} \quad (n,r \geqslant 1). \tag{2.5}$$

Using (2.3) in the formulae

$$Y+iX = -\tfrac{1}{2}\rho\oint\left(\frac{dw}{dz}\right)^2 dz, \tag{2.6}$$

$$M = -\tfrac{1}{2}\rho R\oint z\left(\frac{dw}{dz}\right)^2 dz, \tag{2.7}$$

for the forces (X,Y) and the couple M acting on the cylinder, we find that

$$Y+iX = 2\pi\rho\left\{cA_0+A_0 B_1+i\sum_{n=1}^{\infty}n(n+1)A_n B_{n+1}\right\}, \tag{2.8}$$

$$M = 2\pi\rho Ri\left\{cA_1+\sum_{n=1}^{\infty}n^2A_n B_n\right\}. \tag{2.9}$$

3. It is now necessary to satisfy the boundary condition on the aerofoil. Using (2.1) we may express z^{-r}, $\log z$, z^r in the forms

$$\left.\begin{aligned} z^{-r} = \sum_{n=r}^{\infty}{}^r\beta_n e^{in\zeta}, \qquad \log z = -i\zeta+\sum_{n=0}^{\infty}{}^0\beta_n e^{in\zeta} \\ z^r = \sum_{n=-r}^{\infty}{}^r\gamma_n e^{in\zeta} \end{aligned}\right\}, \tag{3.1}$$

where constant terms, which will not be needed, are ignored. Some of the early coefficients in these expansions are

$$\left.\begin{aligned} &{}^0\beta_1 = \frac{a_1}{a_0}, && {}^0\beta_2 = \frac{a_2}{a_0}-\frac{a_1^2}{2a_0^2}, && {}^0\beta_3 = \frac{a_3}{a_0}-\frac{a_1 a_2}{a_0^2}+\frac{a_1^3}{3a_0^3} \\ &{}^1\beta_1 = \frac{1}{a_0}, && {}^1\beta_2 = -\frac{a_1}{a_0^2}, && {}^1\beta_3 = \frac{a_1^2}{a_0^3}-\frac{a_2}{a_0^2} \\ &{}^2\beta_2 = \frac{1}{a_0^2}, && {}^2\beta_3 = -\frac{2a_1}{a_0^3}, && {}^3\beta_3 = \frac{1}{a_0^3} \\ &{}^1\gamma_1 = a_2, && {}^1\gamma_2 = a_3, && {}^2\gamma_1 = 2a_0 a_3+2a_1 a_2 \\ &{}^1\gamma_{-1} = a_0, && {}^2\gamma_{-1} = 2a_0 a_1, && {}^2\gamma_{-2} = a_0^2 \end{aligned}\right\}, \tag{3.2}$$

and further coefficients can be found if desired. The expansions (3.1) may now be substituted in (2.3) and, assuming that orders of summations may be interchanged, the result can be written

$$w = A_0\zeta + \sum_{n=1}^{\infty} C_n e^{in\zeta} + \sum_{n=1}^{\infty} D_n e^{-in\zeta}, \tag{3.3}$$

where

$$\left.\begin{aligned} C_n &= ca_{n+1} + i\,{}^0\beta_n A_0 + \sum_{r=1}^{n} {}^r\beta_n A_r + \sum_{r=1}^{\infty} {}^r\gamma_n B_r \quad (n \geqslant 1) \\ D_1 &= ca_0 + \sum_{r=1}^{\infty} {}^r\gamma_{-1} B_r \\ D_n &= \sum_{r=n}^{\infty} {}^r\gamma_{-n} B_r \quad (n \geqslant 2) \end{aligned}\right\}. \tag{3.4}$$

The aerofoil $\eta = 0$ is now a stream-line if

$$C_n = \bar{D}_n \quad (n \geqslant 1). \tag{3.5}$$

The next step in the solution is to find the coefficients A_n, and inspection of the equations indicates that it is probably possible to express A_n as a power series in $1/b$. We therefore assume that

$$A_n = \sum_{k=0}^{\infty} {}^nA_k/b^k \quad (n \geqslant 0); \tag{3.6}$$

then, from (2.4),

$$B_n = \sum_{k=0}^{\infty} \Big\{ {}^n\alpha_0\, {}^0A_k + \sum_{r=1}^{\infty} {}^n\alpha_r\, {}^r\bar{A}_k/b^r \Big\} \Big/ b^{n+k}, \tag{3.7}$$

and, assuming that the order of summations in the resulting double and triple series in (3.4) may be changed so that the series may be summed 'diagonally', the constants C_n, D_n now take the forms

$$\begin{aligned} C_n = ca_{n+1} &+ \sum_{k=0}^{\infty} \Big\{ i\,{}^0\beta_n\, {}^0A_k + \sum_{r=1}^{n} {}^r\beta_n\, {}^rA_k \Big\} \Big/ b^k + \\ &+ \sum_{k=1}^{\infty} \sum_{m=1}^{k} {}^m\gamma_n\, {}^m\alpha_0\, {}^0A_{k-m}/b^k + \\ &+ \sum_{k=2}^{\infty} \sum_{r=1}^{k-1} \sum_{m=1}^{r} {}^m\gamma_n\, {}^m\alpha_{k-r}\, {}^{k-r}\bar{A}_{r-m}/b^k \quad (n \geqslant 1), \end{aligned} \tag{3.8}$$

$$\begin{aligned} D_1 = ca_0 &+ \sum_{k=1}^{\infty} \sum_{m=1}^{k} {}^m\gamma_{-1}\, {}^m\alpha_0\, {}^0A_{k-m}/b^k + \\ &+ \sum_{k=2}^{\infty} \sum_{r=1}^{k-1} \sum_{m=1}^{r} {}^m\gamma_{-1}\, {}^m\alpha_{k-r}\, {}^{k-r}\bar{A}_{r-m}/b^k, \end{aligned} \tag{3.9}$$

$$\begin{aligned} D_n = &\sum_{k=n}^{\infty} \sum_{m=n}^{k} {}^m\gamma_{-n}\, {}^m\alpha_0\, {}^0A_{k-m}/b^k + \\ &+ \sum_{k=n+1}^{\infty} \sum_{r=n}^{k-1} \sum_{m=n}^{r} {}^m\gamma_{-n}\, {}^m\alpha_{k-r}\, {}^{k-r}\bar{A}_{r-m}/b^k \quad (n \geqslant 2). \end{aligned} \tag{3.10}$$

The boundary conditions (3.5) must be true for every value of b, so that, using (3.8)–(3.10) and equating coefficients of corresponding powers of $1/b$, we obtain the relations

$$\left.\begin{aligned} ca_2+i\,{}^0\beta_1\,{}^0A_0+{}^1\beta_1\,{}^1A_0 &= c\bar{a}_0 \\ ca_{n+1}+i\,{}^0\beta_n\,{}^0A_0+\sum_{r=1}^{n}{}^r\beta_n\,{}^rA_0 &= 0 \quad (n \geqslant 2) \end{aligned}\right\}, \tag{3.11}$$

$$i\,{}^0\beta_n\,{}^0A_1+\sum_{r=1}^{n}{}^r\beta_n\,{}^rA_1+{}^1\gamma_n\,{}^1\alpha_0\,{}^0A_0 = \begin{cases} {}^1\bar{\gamma}_{-1}\,{}^1\bar{\alpha}_0\,{}^0A_0 & (n=1), \\ 0 & (n>1) \end{cases}, \tag{3.12}$$

$$\left.\begin{aligned} & i\,{}^0\beta_n\,{}^0A_k+\sum_{r=1}^{n}{}^r\beta_n\,{}^rA_k+\sum_{m=1}^{k}{}^m\gamma_n\,{}^m\alpha_0\,{}^0A_{k-m}+ \\ & \qquad\qquad +\sum_{r=1}^{k-1}\sum_{m=1}^{r}{}^m\gamma_n\,{}^m\alpha_{k-r}\,{}^{k-r}\bar{A}_{r-m} \\ & = \begin{cases} \displaystyle\sum_{m=n}^{k}{}^m\bar{\gamma}_{-n}\,{}^m\bar{\alpha}_0\,{}^0A_{k-m}+\sum_{r=n}^{k-1}\sum_{m=n}^{r}{}^m\bar{\gamma}_{-n}\,{}^m\bar{\alpha}_{k-r}\,{}^{k-r}A_{r-m} \\ \qquad\qquad\qquad (k \geqslant 2;\ k>n), \\ {}^k\bar{\gamma}_{-k}\,{}^k\bar{\alpha}_0\,{}^0A_0 \quad (n=k \geqslant 2), \\ 0 \quad (n>k \geqslant 2) \end{cases} \end{aligned}\right\}. \tag{3.13}$$

From these equations the coefficients ${}^nA_k$ for $n \geqslant 1$ can be expressed successively in terms of the circulation coefficients ${}^0A_k$. Six of these coefficients are given below, but, apart from the length of the expressions, further coefficients could readily be obtained. Thus

$$\left.\begin{aligned} {}^1A_0 &= ca_0(\bar{a}_0-a_2)-ia_1\,{}^0A_0 \\ {}^1A_1 &= \tfrac{1}{2}a_0(a_2-\bar{a}_0){}^0A_0-ia_1\,{}^0A_1 \\ {}^1A_2 &= \tfrac{1}{4}ca_0\bar{a}_0(a_0\bar{a}_0-2a_0a_2+a_2\bar{a}_2)-ia_1\,{}^0A_2- \\ &\quad -\tfrac{1}{2}a_0(\bar{a}_0-a_2){}^0A_1+\tfrac{1}{4}ia_0\{a_0a_3+(\bar{a}_0-a_2)(\bar{a}_1-a_1)\}{}^0A_0 \\ {}^2A_0 &= ca_0\{a_1(\bar{a}_0-a_2)-a_0a_3\}-i(a_0a_2+\tfrac{1}{2}a_1^2){}^0A_0 \\ {}^2A_1 &= \tfrac{1}{2}a_0\{a_0a_3-a_1(\bar{a}_0-a_2)\}{}^0A_0-i(a_0a_2+\tfrac{1}{2}a_1^2){}^0A_1 \\ {}^3A_0 &= ca_0\{(\bar{a}_0-a_2)(a_0a_2+a_1^2)-2a_0a_1a_3-a_0^2a_4\}- \\ &\quad -i(a_0^2a_3+2a_0a_1a_2+\tfrac{1}{3}a_1^3){}^0A_0 \end{aligned}\right\}. \tag{3.14}$$

With the help of (3.6) and (2.4) the expression (2.8) for the forces on the cylinder gives $X = 0$ and

$$\frac{Y}{\pi\rho} = 2cA_0-\frac{A_0^2}{b}-2A_0\sum_{r=1}^{\infty}\frac{r}{(2i)^{r+1}b^{r+1}}\{\bar{A}_r+(-)^{r+1}A_r\}- \\ -\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\frac{(n+r)!\,i^{n-r}}{(n-1)!(r-1)!\,2^{n+r}}\frac{A_n\bar{A}_r}{b^{n+r+1}}, \tag{3.15}$$

so that, by writing
$$Y = \sum_{k=0}^{\infty} Y_k/b^k, \tag{3.16}$$
the coefficients Y_k can be found once and for all for any given type of cylinder. The first few coefficients are

$$Y_0/\rho = 2\pi c\, {}^0\!A_0, \tag{3.17}$$

$$Y_1/\rho = 2\pi c\, {}^0\!A_1 - \pi({}^0\!A_0)^2, \tag{3.18}$$

$$\begin{aligned} Y_2/\rho = {} & 2\pi c\, {}^0\!A_2 - 2\pi\, {}^0\!A_0\, {}^0\!A_1 + \tfrac{1}{2}\pi c\{a_0(\bar{a}_0 - a_2) + \bar{a}_0(a_0 - \bar{a}_2)\}{}^0\!A_0 - \\ & - \tfrac{1}{2} i\pi(a_1 - \bar{a}_1)({}^0\!A_0)^2, \end{aligned} \tag{3.19}$$

$$\begin{aligned} Y_3/\rho = {} & -\tfrac{1}{2}\pi c^2 a_0 \bar{a}_0(\bar{a}_0 - a_2)(a_0 - \bar{a}_2) + \\ & + \tfrac{1}{2}\pi i c\{(a_1 - \bar{a}_1)(2a_0\bar{a}_0 - a_0 a_2 - \bar{a}_0\bar{a}_2) - a_0^2 a_3 + \bar{a}_0^2 \bar{a}_3\}{}^0\!A_0 + \\ & + \tfrac{1}{2}\pi c(2a_0\bar{a}_0 - a_0 a_2 - \bar{a}_0\bar{a}_2){}^0\!A_1 + \\ & + \tfrac{1}{4}\pi\{a_1^2 + \bar{a}_1^2 - 2a_1\bar{a}_1 - 2a_0\bar{a}_0 + 3(a_0 a_2 + \bar{a}_0\bar{a}_2)\}({}^0\!A_0)^2 - \\ & - i\pi(a_1 - \bar{a}_1){}^0\!A_0\, {}^0\!A_1 - 2\pi\, {}^0\!A_0\, {}^0\!A_2 - \pi({}^0\!A_1)^2 + 2\pi c\, {}^0\!A_3. \end{aligned} \tag{3.20}$$

The expression (2.9) for the couple can be dealt with in a similar way.

4. To complete the solution of the aerofoil problem it is necessary to obtain a value for the circulation. Adopting the usual procedure for aerofoil sections possessing a sharp trailing edge we choose the circulation so that the stream leaves the trailing edge smoothly. The aerofoil transformation (2.1) can always be chosen so that the trailing edge is the point $\zeta = -\pi$ and the condition for finite velocity at this point is that $dw/d\zeta = 0$ for $\zeta = -\pi$. From (3.3) this gives

$$A_0 + i\sum_{n=1}^{\infty}(-)^n n(\bar{D}_n - D_n) = 0 \tag{4.1}$$

if the condition (3.5) is also used. If the expressions (3.6), (3.9), (3.10) for A_0 and D_n are substituted in (4.1), the resulting equation must be true for all values of b, so that, equating to zero the coefficients of each power of $1/b$, we obtain

$$\left.\begin{aligned} {}^0\!A_0 = {} & ic(\bar{a}_0 - a_0) \\ {}^0\!A_1 = {} & i({}^1\bar{\gamma}_{-1}\, {}^1\bar{\alpha}_0 - {}^1\gamma_{-1}\, {}^1\alpha_0){}^0\!A_0 \\ {}^0\!A_k = {} & i\sum_{n=1}^{k}\sum_{m=n}^{k}(-)^{n+1}n({}^m\bar{\gamma}_{-n}\, {}^m\bar{\alpha}_0\, {}^0\!A_{k-m} - {}^m\gamma_{-n}\, {}^m\alpha_0\, {}^0\!A_{k-m}) + \\ & + i\sum_{n=1}^{k-1}\sum_{r=n}^{k-1}\sum_{m=n}^{r}(-)^{n+1}n({}^m\bar{\gamma}_{-n}\, {}^m\bar{\alpha}_{k-r}\, {}^{k-r}\!A_{r-m} - {}^m\gamma_{-n}\, {}^m\alpha_{k-r}\, {}^{k-r}\!\bar{A}_{r-m}) \\ & \qquad\qquad\qquad\qquad (k \geqslant 2) \end{aligned}\right\}. \tag{4.2}$$

From these equations and (3.14) the circulation coefficients ${}^0\!A_k$ can be obtained successively and explicitly in terms of the constants a_n which define the aerofoil section. Results are given here for the coefficients ${}^0\!A_0$, ${}^0\!A_1$, ${}^0\!A_2$. Expressions for further coefficients are necessarily more lengthy, but there is no theoretical difficulty in finding them, and, for a flat plate in which all the coefficients a_n are zero except a_0 and a_2, the work is relatively simple. Since results for the flat plate have been obtained as far as terms in $1/b^4$ by Tomotika and Havelock, there appears to be no point in reproducing these results here. It should be noticed, however, that the method used here, although having some connexion with Havelock's work, appears to be somewhat simpler in concept.

The coefficients ${}^0\!A_0$, ${}^0\!A_1$, ${}^0\!A_2$ are

$$\left.\begin{array}{c} {}^0\!A_0 = ic(\bar{a}_0-a_0), \qquad {}^0\!A_1 = \tfrac{1}{2}c(\bar{a}_0-a_0)^2 \\ {}^0\!A_2 = \tfrac{1}{4}ic\{a_0\bar{a}_0(\bar{a}_2-a_2)+(\bar{a}_0-a_0)^2(a_1-\bar{a}_1)+3a_0\bar{a}_0(\bar{a}_0-a_0)\} \end{array}\right\}, \tag{4.3}$$

and, from (3.17)–(3.19), the coefficients Y_0, Y_1, Y_2 in the expression (3.16) for the lift are

$$\left.\begin{array}{l} Y_0 = 2\pi i\rho c^2(\bar{a}_0-a_0), \qquad Y_1/Y_0 = i(a_0-\bar{a}_0) \\ \dfrac{Y_2}{Y_0} = \tfrac{1}{2}(\bar{a}_0-a_0)(a_1-\bar{a}_1)+\tfrac{3}{4}a_0\bar{a}_0-\tfrac{1}{2}(\bar{a}_0-a_0)^2+ \\ \qquad\qquad +\dfrac{a_0\bar{a}_0(\bar{a}_2-a_2)}{4(\bar{a}_0-a_0)}+\tfrac{1}{4}(2a_0\bar{a}_0-a_0a_2-\bar{a}_0\bar{a}_2) \end{array}\right\}. \tag{4.4}$$

Applications of these general results are considered in the next section.

5. Fig. 1 shows a circular-arc aerofoil ABC near a plane wall.

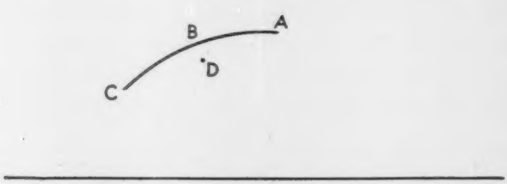

FIG. 1.

The mid-point of the arc is B, and D is the mid-point of the chord AC. Since the theory developed above requires that the origin of

coordinates should be 'inside' the aerofoil, B is a suitable origin and the required transformation for the circular arc is

$$\frac{z+ia(1-e^{2i\alpha})e^{i\theta}}{z+ia(1-e^{-2i\alpha})e^{i\theta}} = \left(\frac{1+e^{i\zeta}}{1-e^{i\zeta-2i\alpha}}\right)^2, \tag{5.1}$$

so that the coefficients in (2.1) for this problem are

$$\left.\begin{aligned} a_0 &= ae^{i(\theta+\alpha)}\sin\alpha \\ a_1 &= -iae^{i\theta}\sin^2\alpha \\ a_n &= -(-i)^n e^{-i(n-1)\alpha}e^{i\theta}\sin^{n-1}\alpha\cos^2\alpha \quad (n>1) \\ l &= 2a\sin 2\alpha \end{aligned}\right\}, \tag{5.2}$$

where θ is the inclination of the chord AC to the wall, 4α is the angle subtended by the arc AC at the centre of its circle, l is the chord AC. The height b of B above the ground is related to the height H of D above the ground by the equation

$$b = H+\tfrac{1}{2}l\tan\alpha\cos\theta. \tag{5.3}$$

From (5.2) and (4.4) we find

$$\left.\begin{aligned} Y_0 &= \pi\rho lc^2\sin(\theta+\alpha)/\cos\alpha \\ Y_1 &= -\tfrac{1}{2}lY_0\sin(\theta+\alpha)/\cos\alpha \\ \frac{Y_2}{l^2Y_0} &= \frac{\sin\theta\sin(\theta+\alpha)}{8\cos\alpha}+\frac{\sin(\theta-\alpha)}{64\sin(\theta+\alpha)}+\frac{5}{64\cos^2\alpha}-\frac{\cos 2\theta}{32} \end{aligned}\right\}. \tag{5.4}$$

For a flat plate Tomotika (**10**) found that, as the distance of the plate from the ground decreases, the value of Y/Y_0 always decreases when θ is large, but, for small θ, Y/Y_0 first decreases and then increases. Calculations by the present writer for a circular-arc aerofoil (**1**) seemed to show that Y/Y_0 always decreases even for small values of θ. The reason for this can be seen by studying equations (5.4). The term Y_1/lY_0 for $\alpha = 0$ and small θ is small, but, as α increases, the effect of this term becomes more marked and would therefore tend to keep Y/Y_0 less than 1 for a greater range of values of l/b. This is best illustrated by taking a numerical example corresponding to the work carried out previously. Thus, if $\alpha = 11°$ and $\theta = 5°$,

$$\frac{Y}{Y_0} = 1-0{\cdot}1404\left(\frac{l}{b}\right)+0{\cdot}04744\left(\frac{l}{b}\right)^2+..., \tag{5.5}$$

and for a flat plate ($\alpha = 0$) the result is

$$\frac{Y}{Y_0} = 1-0{\cdot}04358\left(\frac{l}{b}\right)+0{\cdot}06392\left(\frac{l}{b}\right)^2+.... \tag{5.6}$$

6. The effect of thickness on the results can also be seen by considering the symmetrical aerofoil of small thickness, which corresponds to

$$\left.\begin{array}{c} a_0 = \tfrac{1}{4}l(1+\epsilon)e^{i\theta},\quad a_1 = \tfrac{1}{4}l\epsilon e^{i\theta},\quad a_2 = \tfrac{1}{4}l(1-\epsilon)e^{i\theta},\quad a_3 = -\tfrac{1}{4}l\epsilon e^{i\theta} \\ a_n = 0 \quad (n \geqslant 4) \end{array}\right\}, \tag{6.1}$$

where ϵ is small and real and l is the chord of the aerofoil. For this case

$$\left.\begin{aligned} Y_0 &= \pi\rho lc^2(1+\epsilon)\sin\theta \\ \frac{Y_1}{lY_0} &= -\tfrac{1}{2}(1+\epsilon)\sin\theta \\ \frac{Y_2}{l^2Y_0} &= \tfrac{1}{32}(2+5\epsilon+3\epsilon^2)+\tfrac{3}{16}(1+\epsilon)^2\sin^2\theta \end{aligned}\right\}. \tag{6.2}$$

For small θ the term Y_1/lY_0 is slightly increased numerically by the small thickness, but the effect on the term Y_2/l^2Y_0 is somewhat greater, so that we might expect that the thickness would accentuate the tendency for Y/Y_0 to decrease and then increase as l/b increases. It therefore appears that, for small values of θ, the effects of camber and thickness work mainly in opposite directions so that the flat plate might be expected to give a reasonable approximation to actual cases.

7. In this section the first steps are given for the solution of the problem of the general aerofoil (2.1) in a uniform stream c which is bounded by the plane walls $y = a$, $y = -b$, together with a circulation $K = 2\pi k$ around the aerofoil. If

$$d = a+b, \qquad \alpha = \pi(b-a)/2d, \tag{7.1}$$

a notation used by Havelock (**2**), then a suitable complex potential function w which makes the boundary walls stream-lines is

$$w = cz+ik\left\{\log\sinh\frac{\pi z}{2d}-\log\cosh\left(\frac{\pi z}{2d}+i\alpha\right)\right\}+$$
$$+\sum_{s=1}^{\infty}\frac{(-)^{s-1}}{(s-1)!}\frac{d^s}{dz^s}\left\{A_s\log\sinh\frac{\pi z}{2d}+\bar{A}_s\log\cosh\left(\frac{\pi z}{2d}+i\alpha\right)\right\}. \tag{7.2}$$

Similar functions to those in (7.2) were used by Howland and

McMullen (4) for potential problems relating to groups of circular cylinders. Using the expansions

$$\left.\begin{aligned} \log\sinh\frac{\pi z}{2d} &= \log z+\sum_{n=1}^{\infty}{}^{n}\alpha_0\left(\frac{z}{d}\right)^n \\ \frac{(-)^{s-1}}{(s-1)!}\frac{d^s}{dz^s}\log\sinh\frac{\pi z}{2d} &= \frac{1}{z^s}+\frac{1}{d^s}\sum_{n=1}^{\infty}{}^{n}\alpha_s\left(\frac{z}{d}\right)^n \\ \log\cosh\left(\frac{\pi z}{2d}+i\alpha\right) &= \sum_{n=1}^{\infty}{}^{n}\beta_0\left(\frac{z}{d}\right)^n \\ \frac{(-)^{s-1}}{(s-1)!}\frac{d^s}{dz^s}\log\cosh\left(\frac{\pi z}{2d}+i\alpha\right) &= \frac{1}{d^s}\sum_{n=1}^{\infty}{}^{n}\beta_s\left(\frac{z}{d}\right)^n \end{aligned}\right\}, \tag{7.3}$$

where constant terms which are not needed are omitted, and where

$$\left.\begin{aligned} {}^{2n}\alpha_0 = (-)^{n+1}\frac{\sigma_{2n}}{n}\left(\frac{1}{2}\right)^{2n}, \quad {}^{2n+1}\alpha_0 = 0, \quad \sigma_n = \sum_{k=1}^{\infty}k^{-n} \\ {}^{n}\alpha_s = (-)^{s-1}s\binom{n+s}{n}{}^{n+s}\alpha_0 \\ {}^{n}\beta_0 = \left(\frac{\pi}{2}\right)^n\frac{f_n(\beta)}{n!}, \qquad {}^{n}\beta_s = (-)^{s-1}s\binom{n+s}{s}{}^{n+s}\beta_0 \\ \beta = i\delta = i\tan\alpha, \\ f_{2n}(\beta) = (-)^{n-1}\left\{(1+\delta^2)\frac{d}{d\delta}\right\}^{2n-1}\delta \\ f_{2n+1}(\beta) = i(-)^{n}\left\{(1+\delta^2)\frac{d}{d\delta}\right\}^{2n}\delta \end{aligned}\right\}, \tag{7.4}$$

the complex potential (7.2) may be expressed in the form

$$w = cz+ik\log z+\sum_{n=1}^{\infty}A_n z^{-n}+\sum_{n=1}^{\infty}B_n z^n, \tag{7.5}$$

where
$$B_n = \frac{ik({}^{n}\alpha_0-{}^{n}\beta_0)}{d^n}+\sum_{s=1}^{\infty}\frac{{}^{n}\alpha_s A_s+{}^{n}\beta_s \bar{A}_s}{d^{n+s}}. \tag{7.6}$$

From this point onwards the solution follows that given in § 2 except that results are now expressed as a power series in $1/d$ instead of $1/b$. For a flat plate, terms as far as those given by Tomotika and Havelock can be obtained without much difficulty, and, except for the length of the expressions, there is no theoretical difficulty in carrying out the calculations for the general aerofoil (2.1).

The infinite cascade of similar aerofoils in a uniform stream may also be handled by a similar method, starting from a complex potential function derived from the elementary functions used by Howland (**3**) for potential problems relating to an infinite row of equal circular cylinders.

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NOTE ON AN ASYMPTOTIC FORMULA CONNECTED WITH r-FREE INTEGERS

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LET r, s be positive integers, and $k_1,\dots,k_s$ distinct positive integers. We shall assume throughout that $r \geqslant 2$, and we define $\mu_r(n)$ as 1 or 0 according as n is or is not r-free.*

One of the problems which I considered in a recent paper† was the determination of an asymptotic formula for the sum

$$F(x) = \sum_{0<n\leqslant x} \mu_r(n+k_1)\dots\mu_r(n+k_s).$$

I showed, in particular, that the error term in this asymptotic formula is of the form $O(x^{s/(r+s-1)+\epsilon})$. The object of the present note is to sharpen this result.

Define the symbol $E\begin{pmatrix} n_1,\dots, n_s \\ k_1,\dots, k_s \end{pmatrix}$ as 1 or 0 according as the system of congruences in n

$$n+k_\nu \equiv 0 \pmod{n_\nu} \quad (1 \leqslant \nu \leqslant s) \tag{1}$$

is or is not soluble.

Furthermore, let $\{n_1,\dots,n_s\}$ denote the lowest common multiple of $n_1,\dots,n_s$.

I shall prove the following

THEOREM. *As* $x \to \infty$,

$$F(x) = Ax + O(x^{2/(r+1)+\epsilon}),$$

where the O-constant may depend upon parameters other than x, *and*

$$A = \sum_{a_1,\dots,a_s \geqslant 1} \frac{\mu(a_1)\dots\mu(a_s)}{\{a_1^r,\dots,a_s^r\}} E\begin{pmatrix} a_1^r,\dots, a_s^r \\ k_1,\dots, k_s \end{pmatrix},$$

the infinite series being absolutely convergent.

Remarks. (1) This theorem is only of interest for $s \geqslant 2$, since for $s = 1$ it reduces, in effect, to

$$\sum_{n\leqslant x} \mu_r(n) = \frac{x}{\zeta(r)} + O(x^{2/(r+1)+\epsilon}).$$

* An integer is called r-free if it is not divisible by the rth power of any prime.

† 'Arithmetical pattern problems relating to divisibility by rth powers', in course of publication in *Proc. London Math. Soc.* This paper will be referred to as A.P.

However, the more precise result

$$\sum_{n\leqslant x}\mu_r(n)=\frac{x}{\zeta(r)}+O(x^{1/r})$$

is well known and can be proved in a few lines.

(2) It can be proved (A.P., Lemma 6, case $q=1$) that

$$A=\prod_p\left\{1-\frac{D(p^r|k_1,\ldots,k_s)}{p^r}\right\},$$

where $D(p^r|k_1,\ldots,k_s)$ is defined as the number of integers amongst $1,2,\ldots,p^r$ which are congruent $(\bmod p^r)$ to at least one of $k_1,\ldots,k_s$. This shows, in particular, that $A>0$ provided that, for every prime p,

$$D(p^r|k_1,\ldots,k_s)<p^r,$$

i.e. the numbers $k_1,\ldots,k_s$ do not contain a complete system of residues $(\bmod p^r)$.

On the other hand, if $D(p^r|k_1,\ldots,k_s)=p^r$ for some prime p, then it is easily seen that $F(x)=0$.

(3) We may note that the theorem above enables us to improve the error terms in Theorems 2, 3, and 4 of A.P. to $O(x^{2/(r+1)+\epsilon})$.

Notation. Here x, α, β are positive numbers; ϵ is an arbitrarily small positive number. All other small letters denote positive integers. The O-notation refers to the passage $x\to\infty$, and the O-constants depend at most upon $\epsilon,r,s,k_1,\ldots,k_s,\alpha,\beta$;

$(n_1,\ldots,n_s)$ denotes the highest common factor of $n_1,\ldots,n_s$;

$T\left(x;\begin{matrix}n_1,\ldots,n_s\\k_1,\ldots,k_s\end{matrix}\right)$ denotes the number of systems of numbers $n,l_1,\ldots,l_s$ such that

$$n\leqslant x,$$

$$n+k_\nu=n_\nu l_\nu\quad(1\leqslant\nu\leqslant s);$$

$L(x;k_1,\ldots,k_s;\alpha)$ denotes the number of systems of numbers $n,a_1,\ldots,a_s,b_1,\ldots,b_s$ such that

$$n\leqslant x,$$

$$n+k_\nu=a_\nu^r b_\nu\quad(1\leqslant\nu\leqslant s),$$

$$a_1\ldots a_s>x^\alpha;$$

$d(n)$ is the number of divisors of n.

LEMMA 1. *The system of congruences* (1) *is soluble if and only if*

$$(n_i, n_j)|(k_i-k_j) \quad (1 \leqslant i < j \leqslant s).$$

In the case of solubility the solutions form precisely one residue class

$$(\operatorname{mod}\{n_1,\ldots, n_s\}).$$

This result is well known.*

LEMMA 2. $$T\begin{pmatrix} x; \begin{matrix} n_1,\ldots, n_s \\ k_1,\ldots, k_s \end{matrix} \end{pmatrix} = x\frac{E\begin{pmatrix} n_1,\ldots, n_s \\ k_1,\ldots, k_s \end{pmatrix}}{\{n_1,\ldots, n_s\}} + O(1).$$

This is an almost immediate consequence of Lemma 1.

LEMMA 3. $$\frac{E\begin{pmatrix} n_1,\ldots, n_s \\ k_1,\ldots, k_s \end{pmatrix}}{\{n_1,\ldots, n_s\}} \leqslant \frac{K}{n_1\ldots n_s},$$

where K depends at most upon $s, k_1,\ldots, k_s$.

Proof. We first observe that the assertion is trivial when the system (1) is insoluble. Assume, then, that (1) is soluble. Hence, by Lemma 1,

$$(n_i, n_j) \leqslant |k_i-k_j| \quad (1 \leqslant i < j \leqslant s). \tag{2}$$

Now $n_1\ldots n_s/\{n_1,\ldots, n_s\}$ can be written in the form of a fraction whose numerator and denominator are products of the highest common factors of two or more n_i's at a time. Hence, by (2),

$$\frac{n_1\ldots n_s}{\{n_1,\ldots, n_s\}} \leqslant K,$$

and the lemma follows.

LEMMA 4. $L(x; k_1,\ldots, k_s; \alpha) = O(x^{1-\alpha(r-1)+\epsilon}) + O(x^{2/(r+1)+\epsilon})$.

Proof. The proof is by induction.

$$L(x; k_1; \alpha) = \sum_{\substack{n \leqslant x \\ n+k_1 = a_1^r b_1 \\ a_1 > x^\alpha}} 1 = \sum_{\substack{a_1 \leqslant (x+k_1)^{1/r} \\ a_1 > x^\alpha}} \sum_{\substack{n \leqslant x \\ n+k_1 \equiv 0 (\operatorname{mod} a_1^r)}} 1$$

$$= \sum_{\substack{a_1 \leqslant (x+k_1)^{1/r} \\ a_1 > x^\alpha}} \left\{\frac{x}{a_1^r} + O(1)\right\} = O(x^{1-\alpha(r-1)}) + O(x^{1/r}).$$

Thus the theorem holds for $s = 1$. Next assume that it holds for

* See, e.g., A. Scholz, *Einführung in die Zahlentheorie* (Göschen), Satz 31.

some $s \geqslant 1$. Let β be some number whose value is to be fixed later. Writing $a_1 \ldots a_{s+1} = a$ we have

$$L(x; k_1, \ldots, k_{s+1}; \alpha)$$
$$= O\Big(\sum_{\substack{n \leqslant x \\ n+k_1 = a_1^r b_1 \\ \cdots\cdots\cdots \\ n+k_{s+1} = a_{s+1}^r b_{s+1} \\ a > x^\alpha \\ \frac{a}{a_1}, \ldots, \frac{a}{a_{s+1}} \leqslant x^\beta}} 1\Big) + O\Big(\sum_{\substack{n \leqslant x \\ n+k_1 = a_1^r b_1 \\ \cdots\cdots\cdots \\ n+k_{s+1} = a_{s+1}^r b_{s+1} \\ a > x^\alpha \\ a_1 \ldots a_s > x^\beta}} 1\Big) = L_1 + L_2, \text{ say.}$$

$$L_1 = O\Big(\sum_{\substack{n \leqslant x \\ n+k_1 = a_1^r b_1 \\ \cdots\cdots\cdots \\ n+k_{s+1} = a_{s+1}^r b_{s+1} \\ x^\alpha < a \leqslant x^{\beta(s+1)/s}}} 1\Big) = O\left\{\sum_{x^\alpha < a \leqslant x^{\beta(s+1)/s}} T\left(x; \begin{matrix} a_1^r, \ldots, a_{s+1}^r \\ k_1, \ldots, k_{s+1} \end{matrix}\right)\right\}$$

$$= O\left\{\sum_{x^\alpha < a \leqslant x^{\beta(s+1)/s}} \left(x \frac{E\begin{pmatrix} a_1^r, \ldots, a_{s+1}^r \\ k_1, \ldots, k_{s+1} \end{pmatrix}}{\{a_1^r, \ldots, a_{s+1}^r\}} + 1\right)\right\}, \text{ by Lemma 2,}$$

$$= O\left(x \sum_{a > x^\alpha} \frac{1}{a_1^r \ldots a_{s+1}^r}\right) + O(x^{\beta(s+1)/s+\epsilon}), \text{ by Lemma 3,}$$

$$= O(x^{1-\alpha(r-1)+\epsilon}) + O(x^{\beta(s+1)/s+\epsilon}).$$

$$L_2 = O\Big(\sum_{\substack{n \leqslant x \\ n+k_1 = a_s^r b_1 \\ \cdots\cdots \\ n+k_s = a_s^r b_s \\ a_1 \ldots a_s > x^\beta}} \sum_{a_{s+1}^r b_{s+1} = n+k_{s+1}} 1\Big) = O\Big\{\sum_{\substack{n \leqslant x \\ n+k_1 = a_1^r b_1 \\ \cdots\cdots \\ n+k_s = a_s^r b_s \\ a_1 \ldots a_s > x^\beta}} d(n+k_{s+1})\Big\}$$

$$= O\{x^\epsilon L(x; k_1, \ldots, k_s; \beta)\}$$

$$= O(x^{1-\beta(r-1)+2\epsilon}) + O(x^{2/(r+1)+2\epsilon}), \text{ by assumption.}$$

We therefore have

$$L(x; k_1, \ldots, k_{s+1}; \alpha)$$
$$= O(x^{1-\alpha(r-1)+\epsilon}) + O(x^{2/(r+1)+2\epsilon}) + O(x^{1-\beta(r-1)+2\epsilon}) + O(x^{\beta(s+1)/s+\epsilon}).$$

Putting $\beta = \dfrac{s}{rs+1}$ we obtain

$$L(x; k_1, \ldots, k_{s+1}; \alpha) = O(x^{1-\alpha(r-1)+\epsilon}) + O(x^{2/(r+1)+2\epsilon}),$$

and this proves the lemma.

Proof of the theorem. We begin by noting that the series defining A, namely

$$\sum_{a_1, \ldots, a_s \geqslant 1} \frac{\mu(a_1) \ldots \mu(a_s)}{\{a_1^r, \ldots, a_s^r\}} E\begin{pmatrix} a_1^r, \ldots, a_s^r \\ k_1, \ldots, k_s \end{pmatrix},$$

is absolutely convergent by Lemma 3.

Next, since $\mu_r(n) = \sum_{a^r | n} \mu(a)$, we have

$$F(x) = \sum_{n \leqslant x} \sum_{\substack{a_1^r | n+k_1 \\ \cdots\cdots \\ a_s^r | n+k_s}} \mu(a_1)\ldots\mu(a_s) = \sum_{\substack{n \leqslant x \\ n+k_1 = a_1^r b_1 \\ \cdots\cdots\cdots \\ n+k_s = a_s^r b_s}} \mu(a_1)\ldots\mu(a_s)$$

$$= \sum_{\substack{n \leqslant x \\ n+k_1 = a_1^r b_1 \\ \cdots\cdots\cdots \\ n+k_s = a_s^r b_s \\ a_1 \ldots a_s \leqslant x^{1/r}}} \mu(a_1)\ldots\mu(a_s) + O\left\{L\left(x; k_1,\ldots, k_s; \frac{1}{r}\right)\right\}$$

$$= \sum_{a_1\ldots a_s \leqslant x^{1/r}} \mu(a_1)\ldots\mu(a_s) T\begin{pmatrix} x; & a_1^r,\ldots, a_s^r \\ & k_1,\ldots, k_s \end{pmatrix} + O(x^{2/(r+1)+\epsilon}), \text{ by Lemma 4,}$$

$$= x \sum_{a_1\ldots a_s \leqslant x^{1/r}} \frac{\mu(a_1)\ldots\mu(a_s)}{\{a_1^r,\ldots, a_s^r\}} E\begin{pmatrix} a_1^r,\ldots, a_s^r \\ k_1,\ldots, k_s \end{pmatrix} + O(x^{2/(r+1)+\epsilon}), \text{ by Lemma 2,}$$

$$= Ax + O\left(x \sum_{a_1\ldots a_s > x^{1/r}} \frac{1}{a_1^r\ldots a_s^r}\right) + O(x^{2/(r+1)+\epsilon}), \text{ by Lemma 3,}$$

$$= Ax + O(x^{2/(r+1)+\epsilon}).$$

In conclusion I should like to thank Dr. R. Rado for reading the manuscript and making a number of most helpful suggestions.

A NOTE ON REAL QUADRATIC FORMS

By J. A. TODD (*Cambridge*)

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1. IF A and B are two symmetric $n \times n$ square matrices such that B is non-singular, and if the roots of the equation $|A-\lambda B| = 0$ are all distinct, then the quadratic forms $x'Ax$ and $x'Bx$ can be reduced by the same non-singular linear transformation $x = Ty$ to forms $y'A_1 y$, $y'B_1 y$, where B_1 is the unit matrix and A_1 is a diagonal matrix whose elements are the roots of $|A-\lambda B| = 0$. If these roots are $\lambda_1, \lambda_2,..., \lambda_n$, then the matrix T has for its columns a set of (properly normalized) vectors $\xi^{(i)}$ such that $(A-\lambda_i B)\xi^{(i)} = 0$ $(i = 1, 2,..., n)$, and the proof of the theorem turns on the fact that the n vectors $\xi^{(i)}$ are linearly independent.* If the roots of $|A-\lambda B| = 0$ are not all distinct, such a reduction may or may not be possible. In order that the forms $x'Ax$, $x'Bx$ should be capable of simultaneous reduction to diagonal form, it is necessary (and sufficient) that, corresponding to any p-fold root λ_i of $|A-\lambda B| = 0$, it should be possible to find p linearly independent vectors ξ which satisfy $(A-\lambda_i B)\xi = 0$, or, in other words, that the rank of the matrix $A-\lambda_i B$ should be $n-p$.

A familiar theorem states that, if A and B are *real* and if the form $x'Bx$ is positive definite, then a simultaneous reduction to diagonal form is possible. An essentially equivalent theorem is that any real symmetric matrix A can be reduced to diagonal form by a real *orthogonal* transformation. From this it follows that, if λ_i is a p-fold root of $|A-\lambda I| = 0$, then the rank of $A-\lambda_i I$ is $n-p$. It is difficult to find, in the literature, a simple direct proof of this result, though

* This argument is, I believe, familiar, but a precise reference is not readily accessible. The proof of the theorem given, e.g. in Bôcher, *Introduction to Higher Algebra* (New York, 1924), 167, is inductive. A proof on the lines mentioned above is given, essentially, by Lamb, *Higher Mechanics* (Cambridge, 1920), 220. The very similar problem of reducing the equation of a collineation to diagonal form, in the case of distinct characteristic roots, is worked out on the lines suggested above in Aitken, *Determinants and Matrices* (Edinburgh, 1939), 89, from which the orthogonal reduction of a symmetric matrix is deduced—a problem essentially equivalent to the one considered above.

many references to it can be found.* It is an immediate consequence of the theorem just mentioned, and, conversely, implies this theorem. But the proof of the reduction to diagonal form of a real symmetric matrix by orthogonal transformation given in most elementary texts is inductive in character,† and this proof is open to the aesthetic objection that it does not correspond to the practical method of obtaining the reduction of a given symmetric matrix, which is based on the determination of a set of orthogonal eigen-vectors. The purpose of this note is to provide a simple direct proof of the fact that the rank of $A-\lambda_i I$ is $n-p$ where λ_i is a p-fold root of the real symmetric matrix A. By making use of this result we can then obtain (in an obvious way) an alternative, non-inductive, proof that A can be reduced to a diagonal form by means of an orthogonal transformation, on the same lines as that which is available when the characteristic roots of A are all distinct.

2. The proof depends on the following simple lemma.

LEMMA. *If C is a matrix of rank r, then its r-th compound $C^{(r)}$ is of rank one.*

For, since C is of rank r, there exist non-singular matrices P, Q such that

$$PCQ = D,$$

where D is the matrix in which the first r elements in the principal diagonal are unity and every other element is zero. Hence, by the Binet-Cauchy theorem,‡

$$P^{(r)}C^{(r)}Q^{(r)} = D^{(r)}.$$

But the leading element in $D^{(r)}$ is unity, while every other element is zero. Hence $D^{(r)}$ is of rank one, and so, since $P^{(r)}$ and $Q^{(r)}$ are non-singular, $C^{(r)}$ is also of rank one.

3. We can now prove our main theorem.

THEOREM. *If A is a real symmetric $n \times n$ matrix, and if λ_1 is a p-fold root of the equation $\phi(\lambda) \equiv |A-\lambda I| = 0$, then the matrix $A_1 \equiv A-\lambda_1 I$ is of rank $n-p$.*

* See, e.g., Levi-Civita, *The Absolute Differential Calculus* (London, 1929), 205–7; Hilton, *Homogeneous Linear Substitutions* (Oxford, 1914), 58; Muir, *The Theory of Determinants in the Historical Order of Development*, vol. iii (London, 1920), 294–5.

† See Bôcher, loc. cit., or Ferrar, *Algebra* (Oxford, 1941), 151–3.

‡ See, e.g., Aitken, op. cit. 93–4.

If $\phi^{(k)}(\lambda)$ denotes the kth derivative of $\phi(\lambda)$ with respect to λ, the conditions that λ_1 be a p-fold root of $\phi(\lambda) = 0$ are

$$\phi(\lambda_1) = 0, \qquad \phi^{(k)}(\lambda_1) = 0 \quad (1 \leqslant k < p), \qquad \phi^{(p)}(\lambda_1) \neq 0. \tag{1}$$

By successive application of the rule for differentiating a determinant it is seen that $\phi^{(k)}(\lambda)$ is the sum of all the principal $(n-k)$-rowed minors of $A-\lambda I$, multiplied by the numerical constant $(-1)^k k!$. Hence, using the notation of compound matrices, conditions (1) can be written in the form

$$\text{trace}[A_1^{(n-k)}] = 0 \quad (0 \leqslant k < p), \qquad \text{trace}[A_1^{(n-p)}] \neq 0, \tag{2}$$

since $A_1^{(n)}$ is the 1×1 matrix whose element is $|A_1|$.

From (2) it follows that $A_1^{(n-p)}$ is not the zero matrix. Hence the rank r of A_1 is $n-p$ at least. Suppose, if possible, that $r > n-p$. Denote the matrix $A_1^{(r)}$ by C. Then C is a symmetric matrix (since A_1 is), and is real (since every characteristic root of A is real). If N is the number of rows and columns in C, then, since trace $C = 0$,

$$C_{11}+C_{22}+...+C_{NN} = 0. \tag{3}$$

By the lemma, C is of rank one. Hence, since C is symmetric,

$$C_{rs}^2 = C_{sr}^2 = C_{rs}C_{sr} = C_{rr}C_{ss},$$

and so, from (3),

$$0 = (C_{11}+C_{22}+...+C_{NN})^2 = \sum_{r=1}^{N}\sum_{s=1}^{N} C_{rs}^2.$$

Thus, since C is real, every element of C vanishes, which gives a contradiction. Thus $r = n-p$ and the theorem is established.

THE SIMULTANEOUS REDUCTION OF TWO REAL QUADRATIC FORMS

By W. L. FERRAR (*Oxford*)

[Received 30 March 1947]

1. Introduction

Two real quadratic forms $A(x,x)$ and $C(x,x)$, of which the latter is positive-definite, can be expressed by means of a real non-singular transformation in the forms

$$\sum \lambda_r X_r^2 \quad \text{and} \quad \sum X_r^2.$$

This is commonly proved without finding the actual transformation from x to X. Such a transformation is not difficult to obtain when no two λ's are the same, particularly when $C(x,x)$ is itself merely $\sum x_r^2$. But there appears to be no readily available account of the actual transformation when $C(x,x)$ is the general positive-definite form and the characteristic equation $|A-\lambda C| = 0$ has multiple roots.

The present paper gives the actual transformation under these general conditions. It is, essentially, an amplification and adaptation of the work of Levi-Civita in his *Absolute Differential Calculus*.* Levi-Civita's work is not primarily concerned with a complete treatment of this particular problem, which is only incidental to his main purpose; it leaves out much that is essential to a full treatment of the subject.

I have taken the occasion to use (§ 4) the technique of matrix algebra in place of the more usual methods of carrying out the transformations; once the algebra of the problem has been stated in terms of matrix equations, the actual reduction of $A(x,x)$ and $C(x,x)$ to their canonical forms requires practically no calculation.

2. The rank of the determinant $|A-\lambda C|$

2.1. In the field of real numbers let A, C be symmetrical matrices of order n,

$$A \equiv [a_{ij}], \qquad C \equiv [c_{ij}];$$

let $A(x,x)$, $C(x,x)$ be the corresponding quadratic forms in the n variables $x_1, x_2, \ldots, x_n$; and let $C(x,x)$ be positive-definite. Then the equation

$$|A-\lambda C| = 0$$

* pp. 205–7.

has n roots $\lambda_1, \lambda_2, \dots, \lambda_n$ (necessarily real) and there is a (real) non-singular transformation $x = TX$ such that*

$$A(x,x) \equiv \lambda_1 X_1^2 + \lambda_2 X_2^2 + \dots + \lambda_n X_n^2,$$
$$C(x,x) \equiv X_1^2 + X_2^2 + \dots + X_n^2.$$

By suitable nomenclature we may take any one root of $|A - \lambda C| = 0$ to be λ_1. Let λ_1 be an r-ple root. Then,† for any λ,

$$|a_{ij} - \lambda c_{ij}| = |T|^2 . L(\lambda),$$

where $L(\lambda)$ is the determinant of a diagonal matrix having $\lambda_1 - \lambda$ in the first r places and $\lambda_{r+1} - \lambda, \dots, \lambda_n - \lambda$ in the remaining $n-r$ places.

When $\lambda = \lambda_1$, $L(\lambda) = L(\lambda_1)$, which has precisely r zeros in the diagonal and so is of rank $n-r$. Since T is non-singular‡

$$|a_{ij} - \lambda_1 c_{ij}|$$

is also of rank $n-r$. Hence§ the equations

$$\sum_{i=1}^{n} (a_{ij} - \lambda_1 c_{ij}) \xi_i = 0 \quad (j = 1, 2, \dots, n)$$

have r linearly independent solutions. This establishes the fundamental result on which all the rest of the work depends; *when λ is an r-ple root of $|A - \lambda C| = 0$, the equations*

$$\sum_{i=1}^{n} (a_{ij} - \lambda c_{ij}) \xi_i = 0 \quad (j = 1, 2, \dots, n)$$

have r linearly independent solutions in the ξ_i.

2.2. When $C(x,x) \equiv \sum x_r^2$, this fundamental result can be established by the direct and elegant method discovered by J. A. Todd.‖ It can then be extended to the general positive-definite form $C(x,x)$ by using invariance arguments similar to those given in § 2.1.

3. The 'principal directions' of two quadratic forms

3.1. We use the summation convention** and write our quadratic forms as $\quad A(x,x) \equiv a_{ij} x^i x^j, \qquad C(x,x) \equiv c_{ij} x^i x^j,$

* Ferrar, *Algebra* (Oxford, 1941), 151. Other references to this book will be indicated by F., followed by the page number.

† F. 127, 8. ‡ F. 110. § F. 101.

‖ See the preceding article. The result of § 2.1 is not new, but it is not easy to find, in the literature of the subject, a satisfactory proof that is free of difficult 'invariant-factor' arguments.

** F. 41.

the n variables being x^i $(i = 1, 2, \ldots, n)$. We use the single letter x, when convenient, to denote the n variables x^i. We take $C(x, x)$ to be positive-definite.

We shall find n sets of values

$$x = X_r \quad (r = 1, 2, \ldots, n)$$

for which the ratio

$$\frac{a_{1j} X_r^1 + a_{2j} X_r^2 + \ldots + a_{nj} X_r^n}{c_{1j} X_r^1 + c_{2j} X_r^2 + \ldots + c_{nj} X_r^n}$$

is independent of j, or, on using geometrical language, the vector with components

$$a_{i1} X_r^i, \quad a_{i2} X_r^i, \quad \ldots, \quad a_{in} X_r^i$$

is parallel to the vector with components

$$c_{i1} X_r^i, \quad c_{i2} X_r^i, \quad \ldots, \quad c_{in} X_r^i.$$

For convenience, we call the latter a 'principal direction' vector of the two forms; the vector

$$X_r^1, \quad X_r^2, \quad \ldots, \quad X_r^n$$

from which it is derived we call a 'λ-vector'.

3.2. *Two different roots;* $\lambda_h \neq \lambda_k$

Let λ_h be any root of the characteristic equation

$$|a_{ij} - \lambda c_{ij}| = 0. \tag{1}$$

Then there is a non-zero λ-vector

$$X_h^1, \quad X_h^2, \quad \ldots, \quad X_h^n$$

for which* $\quad (a_{ij} - \lambda_h c_{ij}) X_{(h)}^i = 0 \quad (j = 1, 2, \ldots, n), \qquad (2)$

and, since only ratios of the X are involved in (2), the components can also be made to satisfy

$$c_{ij} X_{(h)}^i X_{(h)}^j = 1. \tag{3}$$

It is at this point that the positive-definite character of C is essential to the argument.

Let $\lambda_k \neq \lambda_h$ be any other root of (1) and let X_k be a corresponding, non-zero, λ-vector. Then

$$(a_{ij} - \lambda_k c_{ij}) X_{(k)}^i = 0. \tag{4}$$

* When a suffix is enclosed in brackets no summation with respect to that suffix is implied, even though the suffix is repeated in the formula.

Multiply this by X_h^j and (2) by X_k^j, sum for $j = 1, 2,..., n$, and subtract the two results; we get, since the matrices A and C are symmetrical,

$$(\lambda_h - \lambda_k)(c_{ij} X_h^i X_k^j) = 0.$$

Since $\lambda_h \neq \lambda_k$,* $$c_{ij} X_h^i X_k^j = 0. \tag{5}$$

3.3. *A multiple root*

Let λ_h be a root of (1) of multiplicity r. Then, by § 2, there are r linearly independent solutions of (2), each of which can be made to satisfy (3); and each of them also satisfies (5) for any $\lambda_k \neq \lambda_h$.

When λ_h is a double root of (1), let

$$X_{h1}^i, \quad X_{h2}^i \quad (i = 1, 2,..., n)$$

be any two independent solutions of the equations (2), each satisfying (3). Then

either $$c_{ij} X_{(h)1}^i X_{(h)2}^j = 0,$$

or we can replace X_{h2} by $X_{h2} + \theta X_{h1}$, choosing θ so that

$$c_{ij} X_{(h)1}^i (X_{(h)2}^j + \theta X_{(h)1}^j) = 0;$$

for in the last equation the coefficient of θ is unity, by (3). We then re-name the solution $X_{h2} + \theta X_{h1}$, calling it, for convenience, X_{h2}.

With this nomenclature,

$$c_{ij} X_{(h)1}^i X_{(h)2}^j = 0. \tag{6}$$

When λ_h is a triple root of (1), there are three linearly independent sets of solutions of (2) and we proceed in the same way. Having chosen the first two sets of X to satisfy (6), either the third set X_{h3} satisfies

$$c_{ij} X_{(h)1}^i X_{(h)3}^j = 0, \qquad c_{ij} X_{(h)2}^i X_{(h)3}^j = 0,$$

or we can replace X_{h3} by

$$X_{h3} + \mu X_{h2} + \nu X_{h1}, \tag{7}$$

choosing μ and ν so that

$$c_{ij} X_{(h)1}^i (X_{(h)3}^j + \mu X_{(h)2}^j + \nu X_{(h)1}^j) = 0,$$

and $$c_{ij} X_{(h)2}^i (X_{(h)3}^j + \mu X_{(h)2}^j + \nu X_{(h)1}^j) = 0.$$

In view of (3) and (6) these reduce to

$$\nu = -c_{ij} X_{(h)1}^i X_{(h)3}^j, \qquad \mu = -c_{ij} X_{(h)2}^i X_{(h)3}^j.$$

Thus, using (if necessary) the notation X_{h3} to represent what we have

* The work of § 3.2 is, of course, not new; all the ideas are familiar and the work is included here so that the proof given may be complete in itself.

previously denoted by (7), we have a set of three linearly independent λ-vectors X_{h1}, X_{h2}, X_{h3} satisfying

$$c_{ij}X^i_{(h)(r)}X^j_{(h)(r)} = 1, \tag{8}$$

$$c_{ij}X^i_{(h)r}X^j_{(h)s} = 0 \quad (r \neq s). \tag{9}$$

Similarly, when the equation (1) has an m-ple root λ_h, we can find m sets satisfying (8) and (9). As we saw at the beginning of this sub-section, each of these also satisfies (5) for any $\lambda_k \neq \lambda_h$.

3.4. *Restatement of the results*

We may sum up in a more convenient form thus. Let $\lambda_1, \lambda_2,..., \lambda_n$ be the n roots of (1). We have found n λ-vectors $X_1, X_2,..., X_n$ satisfying the conditions

$$c_{ij}X^i_{(r)}X^j_{(r)} = 1, \tag{10}$$

$$c_{ij}X^i_rX^j_s = 0 \quad (r \neq s). \tag{11}$$

We may call such a set of λ-vectors a 'λ-frame' and each member of the set a 'frame vector'. To each simple root λ of (1) there corresponds a unique frame-vector (§ 2.1, with $r = 1$). To each m-ple root there corresponds a set of m frame-vectors, which is not unique, but allows an infinity of choice. For example, with a triple root and any given three independent solutions of (2), say

$$X_{h1}, \quad X_{h2}, \quad X_{h3},$$

we may select an arbitrary

$$X_{h1}+\theta X_{h2}+\phi X_{h3}$$

to be our initial X_{h1} of § 3.3. The situation is the familiar one met with in determining principal axes for a conicoid of revolution.

Finally, to each frame vector X_r corresponds one principal-direction vector ξ_r, given by (cf. § 3.1)

$$\xi^i_r = c_{\alpha i}X^\alpha_r \quad (i = 1, 2,..., n). \tag{12}$$

4. The transformation

4.1. We first express the results of § 3 in matrix notation. Indicating a matrix by the element in its rth row and ith column, let

$$A \equiv [a_{ri}], \qquad C \equiv [c_{ri}],$$

$$X \equiv [X^r_i], \qquad \xi \equiv [\xi^i_r].$$

Then the equations (12) of § 3, defining ξ in terms of X, become

$$\xi = X'C, \tag{1}$$

while the 'orthogonal relations', (10) and (11) of § 3, become

$$X'CX = I. \tag{2}$$

From (1) and (2), $\quad \xi X = I, \quad \xi = X^{-1}. \quad$ (3)

Finally, the equations (2) of § 3, satisfied by the λ-vectors, become

$$C'X\Lambda = A'X, \tag{4}$$

where Λ is a diagonal matrix with elements $\lambda_1, \lambda_2,..., \lambda_n$. This results from the following considerations.

When a square matrix is post-multiplied by a diagonal matrix with elements $\lambda_1, \lambda_2,..., \lambda_n$, its ith column is multiplied by λ_i. Accordingly, when

$$C'X = [c_{\alpha r} X_i^\alpha]$$

is post-multiplied by Λ, the product is*

$$[\lambda_{(i)} c_{\alpha r} X_i^\alpha],$$

which, in virtue of equations (2) of § 3, is

$$[a_{\alpha r} X_i^\alpha] = A'X.$$

This proves (4).

4.2. *The transformation of the two forms*

The transformation that reduces $A(x,x)$ and $C(x,x)$ to their canonical forms is

$$y_r = \xi_r^i x^i \quad (r = 1,2,..., n), \tag{5}$$

or, in matrix notation, $\quad y = \xi x, \quad$ (6)

where y and x are single-column matrices with elements $y_1, y_2,..., y_n$ and $x_1, x_2,..., x_n$. This transformation is non-singular, as (3) shows.

Now $y'y$ is a matrix having the single element

$$y_1^2 + y_2^2 + ... + y_n^2.$$

But, by (6) and (1), $\quad y = \xi x = X'Cx, \quad$ (7)

and $\quad y' = x'C'X. \quad$ (8)

Hence $\quad y'y = x'C'X\xi x,$

or, on using (3), $\quad y'y = x'C'x. \quad$ (9)

Again, $y'\Lambda y$ is a matrix having the single element

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2.$$

Now, by (7) and (8),

$$y'\Lambda y = x'C'X\Lambda\xi x = x'A'X\xi x,$$

* The bracket about the suffix i in the formula indicates that there is no summation with respect to i.

on using (4). But $\xi = X^{-1}$, by (3), and so

$$y'\Lambda y = x'A'x. \tag{10}$$

The matrices on the right of (9) and (10) are the matrix expressions for the quadratic forms* $C(x,x)$ and $A(x,x)$. Hence the transformation

$$y_r = \xi_r^i x^i \quad (r = 1, 2, ..., n)$$

yields the identities

$$C(x,x) \equiv y_1^2+y_2^2+...+y_n^2,$$
$$A(x,x) \equiv \lambda_1 y_1^2+\lambda_2 y_2^2+...+\lambda_n y_n^2.$$

* F. 122. The matrices A and C are symmetrical, so that $A' = A$ and $C' = C$. We could, of course, use rows instead of columns in forming the equations at the end of § 2.1.